

Cycle classes and the syntomic regulator

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Abstract

Let $\mathcal{V} = \text{Spec}(R)$ and R be a complete discrete valuation ring of mixed characteristic $(0, p)$. For any flat R -scheme \mathcal{X} we prove the compatibility of the de Rham fundamental class of the generic fiber and the rigid fundamental class of the special fiber. We use this result to construct a syntomic regulator map $\text{reg}_{\text{syn}} : CH^i(\mathcal{X}/\mathcal{V}, 2i - n) \rightarrow H_{\text{syn}}^n(\mathcal{X}, i)$, when \mathcal{X} is smooth over R , with values on the syntomic cohomology defined by A. Besser. Motivated by the previous result we also prove some of the Bloch-Ogus axioms for the syntomic cohomology theory, but viewed as an absolute cohomology theory.

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Introduction

Let $\mathcal{V} = \text{Spec}(R)$, with R a complete discrete valuation ring of mixed characteristic and perfect residue field. Given \mathcal{X} an algebraic \mathcal{V} -scheme one can consider the de Rham cohomology of its generic fiber \mathcal{X}_K and the rigid cohomology of its special fiber \mathcal{X}_k . These two cohomology groups are related (see [BCF04, §6]) by a canonical *cospecialization* map $\text{cosp} : H_{\text{rig}, c}^n(\mathcal{X}_k) \rightarrow H_{\text{dR}, c}^n(\mathcal{X}_K)$ (not an isomorphism in general). There is also a notion of rigid and de Rham cycle class. The starting result of this paper is the compatibility of this cycle classes w.r.t. the cospecialization map (see 1.6 for the precise statement).

In the case \mathcal{X} is smooth (possibly non-proper) over \mathcal{V} we get the following corollary (see Cor. 1.7): let $\text{sp}_{CH} : CH^*(\mathcal{X}_K) \otimes \mathbb{Q} \rightarrow CH^*(\mathcal{X}_k) \otimes \mathbb{Q}$ be the specialization of Chow rings constructed by Grothendieck in [SGA6, Appendix], then the following diagram is commutative

$$\begin{array}{ccc} CH^q(\mathcal{X}_K/K) \otimes \mathbb{Q} & \xrightarrow{\eta_{\text{dR}}} & H_{\text{dR}}^{2q}(\mathcal{X}_K) \\ \text{sp}_{CH} \downarrow & & \downarrow \text{sp} \\ CH^q(\mathcal{X}_k/k) \otimes \mathbb{Q} & \xrightarrow{\eta_{\text{rig}}} & H_{\text{rig}}^{2q}(\mathcal{X}_k/K) \end{array} ,$$

where η_{dR} (resp. η_{rig}) is the de Rham (resp. rigid) cycle class map and sp is the Poincaré dual of cosp .

In the proof we use the main results of [BCF04], [BLR95] and [Pet03].

This result can be viewed as a generalization of a theorem of Messing [GM87, Theorem B3.1] where he further assumes \mathcal{X} to be proper (not only smooth) over \mathcal{V} . In that case rigid cohomology coincides with the crystalline one and the map sp is an isomorphism ([Ber97b]).

This compatibility result is the motivation for an alternative construction of the regulator map (see Prop. 1.13)

$$\text{reg}_{\text{syn}} : CH^i(\mathcal{X}/\mathcal{V}, 2i - n) \rightarrow H_{\text{syn}}^n(\mathcal{X}, i)$$

with values in the syntomic cohomology group defined by Besser in [Bes00] (for \mathcal{X} smooth over \mathcal{V}). For this proof we use an argument of Bloch [Blo86] and the existence of a syntomic cycle class (see Prop. 1.9).

The aforementioned results motivated us to investigate further the properties of the syntomic cohomology. We are not able to formulate even the basic Bloch-Ogus axioms using Besser's framework. Thus we followed Bannai's interpretation of syntomic cohomology as an absolute one [Ban02]. To this aim we define a triangulated category of p -adic Hodge complexes, pHD (see

Def. 2.5). An object M of pHD can be represented by a diagram of the form $M_{\text{rig}} \rightarrow M_K \leftarrow M_{\text{dR}}$ where $M_?$ is a complex of K -vector spaces endowed with a Frobenius automorphism when $? = \text{rig}$ and with a filtration when $? = \text{dR}$. On pHD there is a naturally defined tensor product and we denote the unit object of pHD by \mathbb{K} . The main difference w.r.t. [Ban02] being that the maps in the diagram are not necessarily quasi-isomorphisms.

From [Bes00] we get (in Prop. 5.7) that there are functorial p -adic Hodge complexes $R\Gamma(\mathcal{X})$

$$R\Gamma_{\text{rig}}(\mathcal{X}) \rightarrow R\Gamma_K(\mathcal{X}) \leftarrow R\Gamma_{\text{dR}}(\mathcal{X})$$

inducing the specialization map in cohomology (*i.e.*, taking the cohomology of each element of the diagram). Meanwhile we show how the constructions made by Besser may be obtained using the theory of generalized Godement resolution (also called the bar resolution). In particular we use the results of [vdPS95] in order to have enough points for rigid analytic spaces. Further we consider the twisted version $R\Gamma(\mathcal{X})(i)$, which is given by the same complexes, but with the Frobenius (*resp.* the filtration) twisted by i (see Remark 2.9).

Then we prove (see Prop. 5.10) that the syntomic cohomology groups $H_{\text{syn}}^n(\mathcal{X}, i)$ of [Bes00] are isomorphic to the (absolute cohomology) groups $H_{\text{abs}}^n(\mathcal{X}, i) := \text{Hom}_{pHD}(\mathbb{K}, R\Gamma(\mathcal{X})(i)[n])$. This result generalizes that of [Ban02] (which was given for some particular \mathcal{V} -schemes with good compactification) to any smooth algebraic \mathcal{V} -scheme.

For this absolute cohomology we can prove some of the Bloch-Ogus axioms.

In fact we construct a p -adic Hodge complex $R\Gamma_c(\mathcal{X})(i)$ related to rigid and de Rham cohomology with compact support. So we can define an absolute cohomology with compact support functorial with respect to proper maps $H_{\text{abs},c}^n(\mathcal{X}, i) := \text{Hom}_{pHD}(\mathbb{K}, R\Gamma_c(\mathcal{X})(i)[n])$ and an absolute homology theory $H_n^{\text{abs}}(\mathcal{X}, i) := \text{Hom}_{pHD}(R\Gamma_c(\mathcal{X})(i)[n], \mathbb{K})$.

We want to point out that the above constructions are essentially consequences of the work done by Besser and Bannai, but it seems hard to prove the following results without the formalism of Godement resolutions we develop in § 3: let \mathcal{X} be a smooth scheme over \mathcal{V} , then

- (i) there is a cup product pairing

$$H_{\text{abs}}^n(\mathcal{X}, i) \otimes H_{\text{abs},c}^m(\mathcal{X}, j) \rightarrow H_{\text{abs},c}^{n+m}(\mathcal{X}, i+j)$$

induced by the natural pairings defined on the cohomology of the generic and the special fiber (see Cor. 6.4);

- (ii) there is a Poincaré duality isomorphism (see Prop. 6.5);
- (iii) there is a Gysin map, *i.e.* given a proper morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of smooth algebraic \mathcal{V} -schemes of relative dimension d and e , respectively, then there is a canonical map

$$f_* : H_{\text{abs}}^n(\mathcal{X}, i) \rightarrow H_{\text{abs}}^{n+2c}(\mathcal{Y}, i+c)$$

where $c = e - d$ (see Cor. 6.7).

Notation In this paper R is a complete discrete valuation ring with fraction field K and residue field k , with k perfect. We assume $\text{char}(K) = 0$ and $\text{char}(k) = p > 0$.

We denote by R_0 the ring of Witt vectors of k and K_0 is its field of fractions. The Frobenius of K_0 is denoted by σ .

The category of bounded complexes of K -vector spaces is denoted by $C^b(K)$.

If V is a K -vector space, then V^\vee is the dual vector space.

We use X, Y, \dots for schemes over k or K ; $\mathcal{X}, \mathcal{Y}, \dots$ for K -analytic spaces; \mathbf{P}, \mathbf{Q} for formal K -schemes and $\mathbf{P}_K, \mathbf{Q}_K$ for the associated Raynaud fibers; \mathcal{X}, \mathcal{Y} for (algebraic) \mathcal{V} -schemes, where $\mathcal{V} = \text{Spec}(R)$. Finally, the p -adic completion of a \mathcal{V} -scheme \mathcal{X} will be denoted by $\widehat{\mathcal{X}}$.

1 Cycle classes

- 1.1 (Higher Chow groups). Following Bloch [Blo86] we recall the definition of the higher Chow groups $CH^*(X/K, *)$ of a K -scheme X .

For any $n \leq 0$ let $\Delta^n = \text{Spec}(\mathbb{Z}[t_0, \dots, t_n]/(\sum_i t_i - 1))$ with face maps $\partial_i(n) : \Delta^n \rightarrow \Delta^{n+1}$, which in coordinates are given by $\partial_i(n)(t_0, \dots, t_n) = (t_0, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n)$. Let X be a smooth K -scheme of relative dimension d (this hypothesis is not necessary, but we deal only with smooth schemes). Let $z^q(X/K, 0)$ be the free abelian group generated by the irreducible and closed sub-schemes of X of codimension q . We denote by $z^q(X/K, n)$ the free abelian group generated by the closed sub-schemes $W \subset \Delta_X^n := X \times \Delta^n$ such that $W \in z^q(\Delta_X^n/K, 0)$ and meets all faces properly: *i.e.*, if $F \subset \Delta_X^n$ is a face of codimension c , then the codimension of the irreducible components of the intersection $F \cap W$ is greater than or equal to $c + q$ on Δ_X^n .

Using the differential $\sum_i (-1)^i \partial_i^*(n) : z^q(X/K, n+1) \rightarrow z^q(X/K, n)$ one obtains a complex of abelian groups $z^q(X/K, *)$. We set $\Gamma_X^i(q) := z^q(X/K, 2q - i)$ and

$$CH^q(X/K, 2q - i) := H^i(\Gamma_X(q)) .$$

these groups are in fact isomorphic to the Voevodsky–Suslin motivic cohomology $H_{\text{mot}}^i(X/K, q)$ of the generic fiber X ([MVW06, Theorem 19.1]).

Remark 1.2 (Relative cycles). Let \mathcal{X} be an algebraic and flat \mathcal{V} -scheme. By the theory of relative cycles [SV00] one can define the group $z^q(\mathcal{X}/\mathcal{V}, 0)$ to be the free abelian group generated by *universally integral relative cycles* of codimension q (we can use the codimension because \mathcal{X} is assumed to be equidimensional over \mathcal{V}). By [Ivo05, Part I, Lemma 1.2.6] $z^q(\mathcal{X}/\mathcal{V}, 0)$ is the free abelian group generated by the closed sub-schemes $\mathcal{W} \subset \mathcal{X}$ which are integral, of codimension q and flat over \mathcal{V} . Then we can define the groups $z^q(\mathcal{X}/\mathcal{V}, n)$ as follows: let $z^q(\mathcal{X}/\mathcal{V}, n)$ be the free abelian group generated by the integral and flat \mathcal{V} -schemes $\mathcal{W} \in z^q(\Delta_{\mathcal{X}}^n, 0)$ meeting all faces properly and such that $\partial_i(n-1)^* \mathcal{W}$ is flat over \mathcal{V} for all i . Thus we can form a complex $z^q(\mathcal{X}/\mathcal{V}, *)$ using the same boundary maps of $z^q(X/K, *)$.

Definition 1.3. With the above notation we define the higher Chow groups of \mathcal{X} over \mathcal{V} to be

$$CH^q(\mathcal{X}/\mathcal{V}, 2q - i) := H^i(\Gamma_{\mathcal{X}/\mathcal{V}}(q))$$

where $\Gamma_{\mathcal{X}/\mathcal{V}}^i(q) := z^q(\mathcal{X}/\mathcal{V}, 2q - i)$.

Remark 1.4. (i) Recall that by Lemma 5.1 any closed and flat sub-scheme of \mathcal{X} is completely determined by its generic fiber. Then $z^q(\mathcal{X}/\mathcal{V}, *)$ is a sub-complex of $z^q(\mathcal{X}_K/K, *)$ inducing a canonical map in (co)-homology

$$\gamma : CH^q(\mathcal{X}/\mathcal{V}, 2q - i) := H^i(\Gamma_{\mathcal{X}/\mathcal{V}}(q)) \longrightarrow CH^q(\mathcal{X}_K/K, 2q - i) .$$

(ii) It follows easily by the Snake lemma that, for $i = 2q$, the map $\gamma : CH^q(\mathcal{X}/\mathcal{V}, 0) \rightarrow CH^q(\mathcal{X}_K/K, 0)$ is surjective. In the general case with don't know wether this map is injective or surjective.

- 1.5 (De Rham and rigid fundamental/cycle classes). In the following we refer to [Har75, BCF04, Pet03] for the definitions and the properties of the (algebraic) de Rham and the rigid cohomology theory.

Let W be an integral scheme of dimension r over K (resp. over k). Then we can associate to W its de Rham (resp. rigid) fundamental class which is an element of the dual of the top de Rham (resp. rigid) cohomology with compact support

$$[W]_{\text{dR}} = \text{tr}_{\text{dR}} \in H_{\text{dR}, c}^{2r}(W)^\vee, \text{ resp. } [W]_{\text{rig}} = \text{tr}_{\text{rig}} \in H_{\text{rig}, c}^{2r}(W)^\vee .$$

For the de Rham cohomology this class is first defined in [Har75, §7] as an element of the de Rham homology; by Poincaré duality [BCF04, Theorem 3.4] it corresponds to the trace map. The rigid case is treated in [Pet03, SS2.1, 6].

Now let X be a K -scheme (resp. k -scheme) of dimension d and $w = \sum_i n_i W_i \in z^q(X/K, 0)$ (resp. $\in z^q(X/k, 0)$) be a dimension r cycle on X . The cohomology with compact support is functorial with respect to proper maps, hence there is a canonical map

$$f : \oplus_i H_{\text{rig}, c}^{2r}(W_i)^\vee \rightarrow H_{\text{rig}, c}^{2r}(|w|)^\vee$$

where $|w| = \cup_i W_i$ is the support of w and $? = \text{dR, rig}$ accordingly to the choice of the base field. With the above notations we define the de Rham (resp. rigid) cycle class of w as

$$[w]_? := f \left(\sum_i [W_i]_? \right) \in H_{?,c}^{2r}(|w|)^\vee \quad ? = \text{dR, rig}.$$

Again by functoriality this defines an element of $H_{?,c}^{2r}(X)^\vee$.

Let \mathcal{X} be a flat \mathcal{V} -scheme and let $w \in z^q(\mathcal{X}/\mathcal{V}, 0)$ be a relative cycle. We can write $w = \sum_i n_i \mathcal{W}_i$, where \mathcal{W}_i is an integral flat \mathcal{V} -scheme closed in \mathcal{X} and of codimension q . Then w defines a cycle $w_K \in z^q(\mathcal{X}_K/K, 0)$ (resp. $w_k \in z^q(\mathcal{X}_k/k, 0)$): on the generic fiber we get simply $w_K = \sum_i n_i (\mathcal{W}_i)_K$. However, on the special fiber we must write the irreducible decomposition of each $(\mathcal{W}_i)_k^{\text{red}}$, say $W_{i,1} \cup \dots \cup W_{i,r_i}$, and then consider the multiplicities, i.e.,

$$w_k = \sum_i \left(n_i \sum_j m_{i,j} W_{i,j} \right)$$

where $m_{i,j} := \text{length } \mathcal{O}_{\mathcal{W}_i, W_{i,j}}$.

We can consider the de Rham and the rigid cycle classes of w , i.e.,

$$[w_K]_{\text{dR}} \in H_{\text{dR},c}^{2r}(|w_K|)^\vee, \quad [w_k]_{\text{rig}} \in H_{\text{rig},c}^{2r}(|w_k|)^\vee.$$

Note that the rigid homology groups $H_{2r}^{\text{rig}}(|w_k|)$ are defined as the dual of $H_{\text{rig},c}^{2r}(|w_k|)$ (cf. [Pet03, §2]). We are going to prove that these cycle classes are compatible under (co)-specialization and that they induce a well defined syntomic cohomology class.

Theorem 1.6 (Cycle class compatibility). *Let \mathcal{X} be a flat \mathcal{V} -scheme of relative dimension d and let $w \in z^q(\mathcal{X}/\mathcal{V}, 0)$ be a relative cycle of codimension q (and relative dimension $r = d - q$). Then $\text{cosp}([w_k]_{\text{rig}}) = [w_K]_{\text{dR}}$.*

Proof. First of all consider the basic case: $w = \mathcal{W}$ is an integral closed sub-scheme of \mathcal{X} , smooth over \mathcal{V} . By [BCF04, Proof of theorem 6.9] we have a commutative diagram

$$\begin{array}{ccc} H_{\text{rig},c}^{2r}(\mathcal{W}_k/K) & \xrightarrow{\text{cosp}} & H_{\text{dR},c}^{2r}(\mathcal{W}_K/K) \\ & \searrow [w_k]_{\text{rig}} \quad \swarrow [w_K]_{\text{dR}} & \\ & K & \end{array}$$

where $[w_k]_{\text{rig}}$ (resp. $[w_K]_{\text{dR}}$) is the rigid (resp. de Rham) trace map and it is an isomorphism of K -vector spaces.

Given a general relative cycle w we can reduce to the basic case by arguing as follows. First by linearity and the functoriality of the specialization map we can restrict to the case $w = \mathcal{W}$ with \mathcal{W} integral. Then the generic fiber \mathcal{W}_K is integral and by [EGA4.4, Prop. 17.15.12] there exists a closed K -sub-scheme T such that $\mathcal{W}_K \setminus T$ is smooth over K . Let \mathcal{T} be the flat extension of T (see Lemma 5.1), then \mathcal{T} is of codimension ≥ 1 in \mathcal{W} and the complement $\mathcal{W} \setminus \mathcal{T}$ is a flat \mathcal{V} -scheme of relative dimension r . Consider the long exact sequence

$$\dots H_{\text{dR},c}^{2r-1}(T) \rightarrow H_{\text{dR},c}^{2r}(\mathcal{W}_K \setminus \mathcal{T}_K) \rightarrow H_{\text{dR},c}^{2r}(\mathcal{W}_K) \rightarrow H_{\text{dR},c}^{2r}(T) \dots$$

Note that here the first and last terms vanish for dimensional reasons. The same happens (*mutatis mutandis*) for the rigid cohomology of the special fiber.

Hence from now on we can assume that \mathcal{W} is integral and that its generic fiber \mathcal{W}_K is smooth. In this setting we apply the Reduced Fibre Theorem for schemes [BLR95, Theorem 2.1']: i.e., there exist a finite field extension K'/K and a finite morphism $f : \mathcal{Y} \rightarrow \mathcal{W} \times_{\mathcal{V}} \mathcal{V}'$ of \mathcal{V}' -schemes such that

- (i) $f_{K'} : \mathcal{Y}_{K'} \rightarrow \mathcal{W}_{K'}$ is an isomorphism;
- (ii) \mathcal{Y}/\mathcal{V}' is flat and has reduced geometric fibers.

Recall that the co-specialization map commutes with finite field extension and the same holds for the rigid and the de Rham trace map. By [Pet03, Proof of Prop. 6.4] the rigid fundamental/cycle class is preserved by finite morphisms, *i.e.*,

$$[\mathcal{Y}_{k'}]_{\text{rig}} \circ f_{k'}^* = [\mathcal{W}_{k'}]_{\text{rig}}, \quad f_{k'}^* : H_{\text{rig},c}^{2r}(\mathcal{W}_{k'}) \rightarrow H_{\text{rig},c}^{2r}(\mathcal{Y}_{k'})$$

From the above discussion there is no loss of generality in assuming that \mathcal{W} has reduced geometric fibers and smooth generic fiber \mathcal{W}_k . Now let S be the singular locus of the special fiber \mathcal{W}_k . Again by [EGA4.4, Prop. 17.15.12] the complement $\mathcal{W}_k \setminus S$ is an open and dense sub-scheme of \mathcal{W}_k and it is smooth over k . The scheme S has codimension ≥ 1 in \mathcal{W}_k , hence $H_{\text{rig},c}^{2r}(\mathcal{W}_k \setminus S) \rightarrow H_{\text{rig},c}^{2r}(\mathcal{W}_k)$ is an isomorphism. From this follows that we can assume \mathcal{W} to be smooth over \mathcal{V} where we know that the claim is true. \square

From now on assume \mathcal{X} to be smooth over \mathcal{V} . By the compatibility of (co)-specialization with Poincaré duality [BCF04, Theorem 6.9] the (de Rham or rigid) cycle class map defines an element $\eta_{\text{rig}}(w_k) \in H_{\text{rig},|w_k|}^{2q}(\mathcal{X}_k/K)$ (resp. $\eta_{\text{dR}}(w_K) \in H_{\text{dR},|w_K|}^{2q}(\mathcal{X}_K/K)$) compatible with respect to the specialization morphism

$$sp(\eta_{\text{dR}}(w_K)) = \eta_{\text{rig}}(w_k). \quad (1)$$

Before stating the next corollary we need to introduce some further notation. Let X be a smooth scheme over K (resp. over k), then the de Rham (resp. rigid) cycle class map factors through the Chow groups inducing a map

$$\eta_{\text{dR}} : CH^q(X/K) \rightarrow H_{\text{dR}}^{2q}(X) \quad (\text{resp. } \eta_{\text{rig}} : CH^q(X/k) \rightarrow H_{\text{rig}}^{2q}(X/K))$$

where, by abuse of notation, $\eta_{\text{dR}}(W)$ (resp. $\eta_{\text{rig}}(W)$) is viewed as an element of $H_{\text{dR}}^{2q}(X)$ (resp. $H_{\text{rig}}^{2q}(X/K)$) via the canonical map $H_{\text{dR},W}^{2q}(X) \rightarrow H_{\text{dR}}^{2q}(X)$ (resp. $H_{\text{rig},W}^{2q}(X/K) \rightarrow H_{\text{rig}}^{2q}(X/K)$), for any W integral sub-scheme of codimension q (See [Har75, Prop. 7.8.1 p.60] and [Pet03, Cor. 7.6]).

Also we recall that in [SGA6, Exp. X Appendix] is constructed a specialization map for the classical Chow ring

$$sp_{CH} : CH^*(\mathcal{X}_K/K) \otimes \mathbb{Q} \rightarrow CH^*(\mathcal{X}_k/k) \otimes \mathbb{Q}$$

Explicitly the map is given as follows. Let $W \subset \mathcal{X}_K$ be an integral scheme of codimension q and denote by \mathcal{W} its Zariski closure in \mathcal{X} . Then the specialization of $sp_{CH}[W]$ is the class representing the sub-scheme \mathcal{W}_k .

Corollary 1.7. *With the above notation the following diagram commutes*

$$\begin{array}{ccc} CH^q(\mathcal{X}_K/K) \otimes \mathbb{Q} & \xrightarrow{\eta_{\text{dR}}} & H_{\text{dR}}^{2q}(\mathcal{X}_K) \\ \downarrow sp_{CH} & & \downarrow sp \\ CH^q(\mathcal{X}_k/k) \otimes \mathbb{Q} & \xrightarrow{\eta_{\text{rig}}} & H_{\text{rig}}^{2q}(\mathcal{X}_k/K) \end{array}$$

(we tensored each term by \mathbb{Q} to guarantee the existence of sp_{CH}).

Proof. Straightforward. \square

Let now \mathcal{X} be a smooth \mathcal{V} -scheme. Besser defined the (rigid) syntomic cohomology groups $H_{\text{syn}}^n(\mathcal{X}, i)$ (cf. [Bes00, Def 6.1]). We will be rather sketchy on the definitions because we will give another construction later.

For such a cohomology there is a long exact sequence

$$\cdots \rightarrow H_{\text{syn}}^n(\mathcal{X}, i) \rightarrow H_{\text{rig}}^n(\mathcal{X}_k/K_0) \times F^i H_{\text{dR}}^n(\mathcal{X}_K) \xrightarrow{h} H_{\text{rig}}^n(\mathcal{X}_k/K_0) \times H_{\text{rig}}^n(\mathcal{X}_k/K) \xrightarrow{+} \cdots \quad (2)$$

where $h(x_0, x_{\text{dR}}) = (\phi(x_0^\sigma) - p^i x_0, x_0 \otimes 1_K - sp(x_{\text{dR}}))$.

Roughly these groups are defined as $H_{\text{syn}}^n(\mathcal{X}, i) := H^n(\mathbb{R}\Gamma_{\text{Bes}}(\mathcal{X}, i))$, where

$$\mathbb{R}\Gamma_{\text{Bes}}(\mathcal{X}, i) := \text{Cone}(\mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus \text{Fil}^i \mathbb{R}\Gamma_{\text{dR}}(\mathcal{X}) \rightarrow \mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus \mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}/K))[-1]$$

is a complex of abelian groups functorial in \mathcal{X} and such that

$$H^n(\mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}/K_0)) = H_{\text{rig}}^n(\mathcal{X}_k/K_0), \quad H^n(\text{Fil}^i \mathbb{R}\Gamma_{\text{dR}}(\mathcal{X})) = H_{\text{dR}}^n(\mathcal{X}_K).$$

The functoriality of $\mathbb{R}\Gamma_{\text{Bes}}(-, i)$ allows us to give the following definition.

Definition 1.8. Let \mathcal{X} be a smooth \mathcal{V} -scheme. Let $\mathcal{Z} \subset \mathcal{X}$ be a closed sub-scheme of \mathcal{X} . We define the syntomic complex with support in \mathcal{Z} using the complexes defined by Besser in the following way

$$\mathbb{R}\Gamma_{\text{Bes}, \mathcal{Z}}(\mathcal{X}, i) := \text{Cone}(\mathbb{R}\Gamma_{\text{Bes}}(\mathcal{X}, i) \rightarrow \mathbb{R}\Gamma_{\text{Bes}}(\mathcal{X} \setminus \mathcal{Z}, i))[-1]$$

This is a complex of abelian groups functorial with respect to cartesian squares. This fact will be used in the proof of Prop. 1.13.

The cohomology of this complex is the *syntomic cohomology with support* in \mathcal{Z} , denoted by

$$H_{\text{syn}, \mathcal{Z}}^n(\mathcal{X}, i) := H^n(\mathbb{R}\Gamma_{\text{Bes}, \mathcal{Z}}(\mathcal{X}, i))$$

so that we get a long exact sequence

$$\cdots H_{\text{syn}, \mathcal{Z}}^n(\mathcal{X}, i) \rightarrow H_{\text{syn}}^n(\mathcal{X}, i) \rightarrow H_{\text{syn}}^n(\mathcal{X} \setminus \mathcal{Z}, i) \rightarrow \cdots$$

Proposition 1.9 (Syntomic cycle class). *Let \mathcal{X} be a smooth \mathcal{V} -scheme let $w \in z^q(\mathcal{X}/\mathcal{V}, 0)$ be a relative cycle of codimension q . Let $\psi : H_{\text{syn}, |w|}^{2q}(\mathcal{X}, q) \rightarrow H_{\text{rig}, |w_k|}^{2q}(\mathcal{X}_k/K_0) \times F^q H_{\text{dR}, |w_K|}^{2q}(\mathcal{X}_K/K)$ be the canonical map. Then ψ is injective and there exists a unique element $\eta_{\text{syn}}(w) \in H_{\text{syn}, |w|}^{2q}(\mathcal{X}, q)$ such that $\psi(\eta_{\text{syn}}(w)) = (\eta_{\text{rig}}(w_k), \eta_{\text{dR}}(w_K))$.*

Proof. By the definition of syntomic cohomology with support there is a long exact sequence similar to (2)

$$\begin{aligned} \cdots \rightarrow H_{\text{rig}, |w_k|}^{2q-1}(\mathcal{X}_k/K_0) \oplus H_{\text{rig}, |w_k|}^{2q-1}(\mathcal{X}_k/K) \rightarrow H_{\text{syn}, |w|}^{2q}(\mathcal{X}, q) \rightarrow \\ H_{\text{rig}, |w_k|}^{2q}(\mathcal{X}_k/K_0) \times F^q H_{\text{dR}, |w_K|}^{2q}(\mathcal{X}_K/K) \rightarrow H_{\text{rig}, |w_k|}^{2q}(\mathcal{X}_k/K_0) \oplus H_{\text{rig}, |w_k|}^{2q}(\mathcal{X}_k/K) \rightarrow \cdots \end{aligned}$$

The last term on the left vanishes because of weak purity in rigid cohomology [Ber97b, Cor. 5.7]. It follows that $H_{\text{syn}, |w|}^{2q}(\mathcal{X}, q)$ consists of the pairs $(x, y) \in H_{\text{rig}, |w_k|}^{2q}(\mathcal{X}_k/K_0) \times F^q H_{\text{dR}, |w_K|}^{2q}(\mathcal{X}_K/K)$ such that $\phi(x^\sigma) = p^q x$ and $sp(y) = x \otimes 1_K$. By Hodge theory $F^q H_{\text{dR}, |w_K|}^{2q}(\mathcal{X}_K/K) = H_{\text{dR}, |w_K|}^{2q}(\mathcal{X}_K/K)$. Moreover the Frobenius acts on $[w_k]_{\text{rig}}$ by multiplication for p^q (see [Pet03, Prop. 7.13]). Hence we can easily conclude the proof by (1). \square

Lemma 1.10 (Functoriality). *Let $f : \mathcal{X}' \rightarrow \mathcal{X}$ be a closed immersion of smooth \mathcal{V} -schemes and let $w \in z^q(\mathcal{X}/\mathcal{V}, 0)$ be a relative cycle of codimension q . Assume that the pre-image $f^{-1}w$ lies in $z^q(\mathcal{X}'/\mathcal{V}, 0)$, then $f^* \eta_{\text{syn}}(w) = \eta_{\text{syn}}(f^{-1}w)$.*

Proof. It is not restrictive to assume that $w = \mathcal{W}$ is an integral sub-scheme of \mathcal{X} flat over \mathcal{V} .

We first show that is sufficient to prove that $f^* \eta_{\text{dR}}(w_K) = \eta_{\text{dR}}(f^{-1}w_K)$. In fact the syntomic cycle class can be viewed as an element $(\eta_{\text{rig}}(w_k), \eta_{\text{dR}}(w_K))$ in the direct sum of rigid and de Rham cohomology. Then note that $sp(\eta_{\text{dR}}(w_K)) = \eta_{\text{rig}}(w_k) \otimes 1_K$ and that the specialization map is functorial.

To prove that $f^* \eta_{\text{dR}}(\mathcal{W}_K) = \eta_{\text{dR}}(f^{-1}\mathcal{W}_K)$ we first reduce by excision to the case where \mathcal{W}_K is smooth over K (just remove from \mathcal{W}_K , \mathcal{X}_K and \mathcal{X}'_K the singular points of \mathcal{W}_K). In the same way we can further assume $f^{-1}\mathcal{W}_K$ to be smooth over K . Now we can follow the same proof of [Pet03, Prop. 7.1] to conclude. \square

Lemma 1.11 (Homotopy). *The (rigid) syntomic cohomology is homotopy invariant, i.e.*

$$H_{\text{syn}}^n(\mathcal{X} \times \mathbb{A}_{\mathcal{V}}^1, q) \cong H_{\text{syn}}^n(\mathcal{X}, q).$$

Proof. Just consider the long exact sequences of syntomic cohomology and note that the de Rham cohomology (of smooth schemes) is homotopy invariant by [Har75, Prop. 7.9.1]. The same holds for rigid cohomology, for instance using the Künneth formula [Ber97a]. \square

Lemma 1.12 (Weak purity). *Let \mathcal{X} be a smooth \mathcal{V} -scheme. Let $\mathcal{Z} \subset \mathcal{X}$ be a closed sub-scheme of \mathcal{X} of codimension q . Then*

$$H_{\text{syn}, \mathcal{Z}}^n(\mathcal{X}, i) = 0 \quad \text{for all } n < 2q.$$

Proof. This follows directly from the long exact sequence of syntomic cohomology and the weak purity in de Rham and rigid cohomology ([Har75, §7.2] and [Pet03, §1]). \square

Proposition 1.13. *Let \mathcal{X} be a smooth \mathcal{V} -scheme. The syntomic cycle class map induces a group homomorphism $\text{reg}_{\text{syn}} : CH^i(\mathcal{X}/\mathcal{V}, 2i - n) \rightarrow H_{\text{syn}}^n(\mathcal{X}, i)$.*

Proof. The construction is analogous to that provided in [Blo86]. Consider the cohomological double complex $\mathbb{R}\Gamma_{\text{Bes}}(\Delta_{\mathcal{X}}^{-n}, q)^m$, non-zero for $m \geq 0, n \leq 0$; the differential in n is induced by ∂_i^n in the usual way. Similarly define the double complex

$$\mathbb{R}\Gamma_{\text{Bes}, \text{supp}}(\Delta_{\mathcal{X}}^{-n}, q)^m := \text{colim}_{w \in \tau^q(\mathcal{X}, -n)} \mathbb{R}\Gamma_{\text{Bes}, |w|}(\Delta_{\mathcal{X}}^{-n}, q)^m.$$

For technical reasons we truncate these complex (non-trivially)

$$A_{\mathcal{X}}^{n,m} = \tau_{n \geq -N} \mathbb{R}\Gamma_{\text{Bes}, ?}(\Delta_{\mathcal{X}}^{-n}, q)^m \quad ? = \emptyset, \text{ supp}$$

for N even and $N \gg 0$.

Consider the spectral sequence

$$E_1^{n,m} := H^m(A^{\bullet, n}) \Rightarrow H^{n+m}(sA^{\bullet, *})$$

where s denotes the associated simple complex of a double complex. By homotopy invariance (Lemma 1.11) $E_1^{n,m} := H_{\text{syn}}^m(\Delta_{\mathcal{X}}^{-n}, q)$ is isomorphic to $H_{\text{syn}}^m(\mathcal{X}, q)$ for $-N \leq n \leq 0$ and $m \geq 0$; otherwise it is zero. Moreover $d_1^{n,m} = 0$ except for n even, $-N \leq n < 0$ and $m \geq 0$ in which case $d_1^{n,m} = \text{id}$. This gives an isomorphism $H^i(sA^{n,m}) \cong H_{\text{syn}}^i(\mathcal{X}, q)$.

In the spectral sequence

$$E_{1, \text{supp}}^{n,m} := H^m(A_{\text{supp}}^{\bullet, n}) \Rightarrow H^{n+m}(sA_{\text{supp}}^{\bullet, *})$$

we have

$$E_{1, \text{supp}}^{n,m} = \text{colim}_{w \in \tau^q(\mathcal{X}, -n)} H_{\text{syn}, |w|}^m(\Delta_{\mathcal{X}}^{-n}, q) \quad \text{for } -N \leq n \leq 0, m \geq 0,$$

and $E_{1, \text{supp}}^{n,m} = 0$ otherwise. Applying lemma 1.10 to the face morphisms it is easy to prove that the syntomic cycle class induces a natural map of complexes $\Gamma_{\mathcal{X}/\mathcal{V}}^{\bullet}(q) \rightarrow E_{1, \text{supp}}^{\bullet, -2q, 2q}$ and hence for all i a map $CH^q(\mathcal{X}/\mathcal{V}, 2q - i) \rightarrow E_{2, \text{supp}}^{i-2q, 2q}$. Due to weak purity the groups $E_{r, \text{supp}}^{n,m}$ are zero for $m < 2q$ and $r \geq 1$. Hence there are natural maps $E_{2, \text{supp}}^{i-2q, 2q} \rightarrow E_{\infty, \text{supp}}^{i-2q, 2q} \rightarrow H^i(sA_{\text{supp}}^{\bullet, *})$. By construction there is map $H^i(sA_{\text{supp}}^{\bullet, *}) \rightarrow H^i(sA^{\bullet, *})$. Composing all these maps we obtain the expected map $CH^q(\mathcal{X}, 2q - i) \rightarrow H_{\text{syn}}^i(\mathcal{X}, q)$. \square

Corollary 1.14. *With the above notation there is a commutative diagram*

$$\begin{array}{ccccc} CH^q(\mathcal{X}/\mathcal{V}) \otimes \mathbb{Q} & \xrightarrow{\gamma \otimes 1_{\mathbb{Q}}} & CH^q(\mathcal{X}_K/K) \otimes \mathbb{Q} & \xrightarrow{sp_{CH}} & CH^q(\mathcal{X}_K/k) \otimes \mathbb{Q} \\ \text{reg}_{\text{syn}} \downarrow & & \eta_{\text{dR}} \downarrow & & \eta_{\text{rig}} \downarrow \\ H_{\text{syn}}^{2q}(\mathcal{X}, q) & \xrightarrow{\pi} & F^q H_{\text{dR}}^{2q}(\mathcal{X}_K) & \xrightarrow{sp} & H_{\text{rig}}^{2q}(\mathcal{X}_K/K) \end{array}$$

where π is the composition $H_{\text{syn}}^n(\mathcal{X}, i) \rightarrow H_{\text{rig}}^n(\mathcal{X}_K/K_0) \times F^i H_{\text{dR}}^n(\mathcal{X}_K) \xrightarrow{\text{pr}_2} F^i H_{\text{dR}}^n(\mathcal{X}_K)$; γ is the map described in the remark 1.4.

Proof. Just note that in this case reg_{syn} is the map induced by the syntomic cycle class in the usual way. \square

2 p -adic Hodge complexes

Having defined a regulator map with values into the syntomic cohomology is tempting to check (some of) the Bloch-Ogus axioms for this theory. We address this problem by viewing the syntomic cohomology as an absolute cohomology theory.

Thus in this section we define a triangulated category of p -adic Hodge complexes similar to that of [Ban02]. See also [Beĭ86], [Hub95] and [Lev98, Ch. V §2.3].

The syntomic cohomology will be computed by Hom groups in this category.

Definition 2.1 (cf. [Ban02, 2.1]). Let $C_{\text{rig}}^b(K)$ be the category of pairs (M^\bullet, ϕ) , where

- (i) $M^\bullet = M_0^\bullet \otimes_{K_0} K$ where M_0^\bullet is a complex in $C^b(K_0)$;
- (ii) (Frobenius structure) Let $(M_0^\bullet)^\sigma := M_0^\bullet \otimes_{\sigma} K_0$. Then $\phi : (M_0^\bullet)^\sigma \rightarrow M_0^\bullet$ is a K_0 -linear morphism.

The morphisms in this category are morphisms in $C^b(K_0)$ compatible with respect to the Frobenius structure. In this way we get an abelian category.

Definition 2.2. Let Filt_K be the category of K -vector spaces with a descending, exhaustive and separated filtration. We denote $C_{\text{dR}}^b(K) = C^b(\text{Filt}_K)$ and we write the objects of this category as pairs (M^\bullet, F) , where

- (i) M^\bullet is a complex in $C^b(K)$;
- (ii) (Hodge filtration) F is a (separated and exhaustive) filtration on M^\bullet .

Remark 2.3 (Strictness). We review some technical facts about filtered categories. For a full discussion see [Hub95, § 2 and 3].

The category Filt_K (and also $C_{\text{dR}}^b(K)$) is additive but not abelian. It is an exact category by taking for short exact sequences that which are exact as sequences of K -vector spaces and such that the morphisms are strict w.r.t. the filtrations: recall that a morphism $f : (M, F) \rightarrow (N, F)$ is *strict* if $f(F^i M) = F^i N \cap \text{Im}(f)$.

An object $(M^\bullet, F) \in C_{\text{dR}}^b(K)$ is a *strict complex* if its differentials are strict as morphism of filtered vector spaces. Strict complexes can be characterized also by the fact that the canonical spectral sequence $H^q(F^p) \Rightarrow H^{p+q}(M^\bullet)$ degenerates at E_1 .

One can define canonical truncation functors on $C_{\text{dR}}^b(K)$: for $M^\bullet \in C_{\text{dR}}^b(K)$ let

$$\tau_{\leq n}(M^\bullet, F)^i := \begin{cases} M^i & i < n \\ \text{Ker}(d^n) & i = n \\ 0 & i > n \end{cases} \quad \tau_{\geq n}(M^\bullet, F)^i := \begin{cases} 0 & i < n-1 \\ \text{Coim}(d^n) & i = n-1 \\ M^i & i \geq n \end{cases}.$$

It is important to note that for a strict complex (M^\bullet, F) the naive cohomology object $\tau_{\leq n} \tau_{\geq n}(M^\bullet, F)$ agrees with the cohomology $H^n(M^\bullet)$ of the complex of K -vector spaces underlying (M^\bullet, F) (see [Hub95, Prop. 2.1.3 and §3]).

Remark 2.4 (Comparison/gluing functors). There is an exact functor $\Phi_{\text{rig}} : C_{\text{rig}}^b(K) \rightarrow C^b(K)$ (resp. $\Phi_{\text{dR}} : C_{\text{dR}}^b(K) \rightarrow C^b(K)$) induced by $(M^\bullet, \phi) \mapsto M^\bullet$ (resp. $(M^\bullet, F) \mapsto M^\bullet$).

Definition 2.5 (cf. [Ban02, Def. 2.2]). We define the category pHC of p -adic Hodge complexes whose objects are systems $M = (M_{\text{rig}}^\bullet, M_{\text{dR}}^\bullet, M_K^\bullet, c, s)$, where

- (i) $(M_{\text{rig}}^\bullet, \phi)$ is an object of $C_{\text{rig}}^b(K)$ and $H^*(M_{\text{rig}}^\bullet)$ is finitely generated over K ;
- (ii) $(M_{\text{dR}}^\bullet, F)$ is an object of $C_{\text{dR}}^b(K)$ and $H^*(M_{\text{dR}}^\bullet)$ is finitely generated over K ;
- (iii) M_K^\bullet is an object of $C^b(K)$ and $c : M_{\text{rig}}^\bullet \rightarrow M_K^\bullet$ (resp. $s : M_{\text{dR}}^\bullet \rightarrow M_K^\bullet$) is morphism in $C^b(K)$.

Hence c, s give a diagram in C_K^b

$$M_{\text{rig}}^\bullet \xrightarrow{c} M_K^\bullet \xleftarrow{s} M_{\text{dR}}^\bullet.$$

A morphism in pHC is given by a system $f = (f_{\text{rig}}, f_{\text{dR}}, f_K)$ where $f_? : M_?^\bullet \rightarrow N_?^\bullet$ is a morphism in $C_{\text{rig}}^b(K)$, $C_{\text{dR}}^b(K)$, $C^b(K)$, for $? = \text{rig}, \text{dR}, K$ respectively, and such that they are compatible with respect to the diagram in (iii) above.

A homotopy in pHC is a system of homotopies h_i compatible with the comparison maps c, s . We define the category pHK to be the category pHC modulo morphisms homotopic to zero. We say that a morphism $f = (f_{\text{rig}}, f_{\text{dR}}, f_K)$ in pHC (or pHK) is a *quasi-isomorphism* if: f_{rig} is a quasi-isomorphism for $? = \text{rig}, K$; f_{dR} is a filtered quasi-isomorphism, i.e. $\text{gr}_F(f_{\text{dR}})$ is a quasi-isomorphism. Finally we say that $M \in pHC$ (or pHK) is *acyclic* if $M_{\text{?}} = 0$ is acyclic for any $? = \text{rig}, \text{dR}, K$.

Lemma 2.6. (i) The category pHK is a triangulated category.

(ii) The localization of pHK with respect to the class of quasi-isomorphisms exists. This category, denoted by pHD , is a triangulated category.

(iii) On the category pHD it is possible to define a non-degenerate t -structure (resp. truncation functors) compatible with the standard t -structure (resp. truncation functors) defined on $C^b(K)$, $C_{\text{rig}}^b(K)$ and $C_{\text{dR}}^b(K)$.

Proof. The proof is the same as [Ban02, Prop. 2.6]. See also [Hub95, §2] for a survey on how to derive exact categories. \square

Remark 2.7. In the terminology of Huber [Hub95, §4] the category pHC is the *glued exact category* of $C_{\text{rig}}^b, C_{\text{dR}}^b$ via C_K^b . The foregoing definition is inspired by [Ban02] where Bannai constructs a *rigid glued exact category* C_{MF}^b , i.e., the comparison maps are all quasi-isomorphisms. For our purposes we cannot impose this strong assumption. This is motivated by the fact that the (co)-specialization is not an isomorphism for a general smooth \mathcal{V} -scheme.

- 2.8 (Notation). Let M^\bullet, M'^\bullet be two objects of pHC . Consider the following diagram $\mathcal{H}(M^\bullet, M'^\bullet)$ (of complexes of abelian groups)

$$\begin{array}{ccccc} \text{Hom}_{K_0}^\bullet((M_0^\bullet)^\sigma, M_0'^\bullet) & & \text{Hom}_K^\bullet(M_{\text{rig}}^\bullet, M_K'^\bullet) & & \text{Hom}_K^\bullet(M_{\text{dR}}^\bullet, M_K'^\bullet) \\ & \nwarrow h_0 & \nearrow h_1 & \nwarrow h_2 & \nearrow h_3 \\ & \text{Hom}_{K_0}^\bullet(M_0^\bullet, M_0'^\bullet) & & \text{Hom}_K^\bullet(M_K^\bullet, M_K'^\bullet) & & \text{Hom}_K^{\bullet, F}(M_{\text{dR}}^\bullet, M_{\text{dR}}'^\bullet) \\ & & & \nearrow h_4 & \end{array}$$

where $h_0(x_0) = x_0 \circ \phi - \phi' \circ x_0^\sigma$, $h_1(x_0) = c' \circ (x_0 \otimes \text{id}_K)$, $h_2(x_K) = x_K \circ c$, $h_3(x_K) = x_K \circ s$, $h_4(x_{\text{dR}}) = s' \circ x_{\text{dR}}$. Then define the complexes of abelian groups $\Gamma_0(M^\bullet, M'^\bullet) := \text{direct sum of the bottom row}$, $\Gamma_1(M^\bullet, M'^\bullet) := \text{direct sum of the top row}$. Consider the morphism

$$\psi_{M^\bullet, M'^\bullet} : \Gamma_0(M^\bullet, M'^\bullet) \rightarrow \Gamma_1(M^\bullet, M'^\bullet) \quad (x_0, x_K, x_{\text{dR}}) \mapsto (-h_0(x_0), h_1(x_0) - h_2(x_K), h_3(x_K) - h_4(x_{\text{dR}}))$$

Finally let $\Gamma(M^\bullet, M'^\bullet) := \text{Cone}(\psi_{M^\bullet, M'^\bullet})[-1]$.

Remark 2.9. (i) Let $\mathbb{K}(-n)$ be the Tate twist p -adic Hodge complex: $\mathbb{K}(-n)_{\text{rig}}$ (resp. $\mathbb{K}(-n)_{\text{dR}}, \mathbb{K}(-n)$) is equal to K concentrated in degree zero; the Frobenius is $\phi(\lambda) := p^n \sigma(\lambda)$; the filtration is $F^i = K$ for $i \leq n$ and zero otherwise.

(ii) Given two p -adic Hodge complexes M^\bullet and M'^\bullet we define their tensor product $M^\bullet \otimes M'^\bullet$ component-wise, i.e., $(M_{\text{rig}}^\bullet \otimes M_{\text{rig}}'^\bullet, M_{\text{dR}}^\bullet \otimes M_{\text{dR}}'^\bullet, M_K^\bullet \otimes M_K'^\bullet, c \otimes c', s \otimes s')$. We denote by $M^\bullet(n)$ the complex $M^\bullet \otimes \mathbb{K}(n)$.

(iii) The complex $\Gamma(\mathbb{K}, M^\bullet(n))$ is quasi-isomorphic to

$$\text{Cone}(M_0^\bullet \oplus F^n M_{\text{dR}}^\bullet \xrightarrow{\eta} M_0^\bullet \oplus M_K^\bullet)[-1] \quad \eta(x_0, x_{\text{dR}}) = (p^{-n} \phi(x_0^\sigma) - x_0, c(x_0 \otimes \text{id}_K) - s(x_{\text{dR}}))$$

where $x_0 \in M_0^\bullet$, $x_{\text{dR}} \in F^n M_{\text{dR}}^\bullet$.

If c is a quasi-isomorphism, letting sp denote the composition of

$$H^q(F^n M_{\text{dR}}^\bullet) \xrightarrow{s^*} H^q(M_K^\bullet) \xleftarrow{c^*} H^q(M_{\text{rig}}^\bullet),$$

we obtain a long exact sequence

$$\cdots \rightarrow H^q(\Gamma(\mathbb{K}, M^\bullet(n))) \rightarrow H^q(M_0^\bullet) \oplus H^q(F^n M_{\text{dR}}^\bullet) \xrightarrow{\eta'} H^q(M_0^\bullet) \oplus H^q(M_{\text{rig}}^\bullet) \rightarrow \cdots,$$

where $\eta'(x_0, x_{\text{dR}}) = (p^{-n}\phi(x_0^\sigma) - x_0, x_0 \otimes 1_K - sp(x_{\text{dR}}))$.

If s is a quasi-isomorphism, letting cosp denote the composition of

$$H^q(M_{\text{rig}}^\bullet) \xrightarrow{c^*} H^q(M_K^\bullet) \xleftarrow{s^*} H^q(M_{\text{dR}}^\bullet),$$

we obtain a long exact sequence

$$\cdots \rightarrow H^q(\Gamma(\mathbb{K}, M^\bullet(n))) \rightarrow H^q(M_0^\bullet) \oplus H^q(F^n M_{\text{dR}}^\bullet) \xrightarrow{\eta''} H^q(M_0^\bullet) \oplus H^q(M_{\text{dR}}^\bullet) \rightarrow \cdots,$$

where $\eta''(x_0, x_{\text{dR}}) = (p^{-n}\phi(x_0^\sigma) - x_0, \text{cosp}(x_0 \otimes 1_K) - x_{\text{dR}})$.

Proposition 2.10 (Ext-formula). *With the above notation there is a canonical morphism of abelian groups*

$$\text{Hom}_{pHD}(M^\bullet, M'^\bullet[n]) \cong H^n(\Gamma(M^\bullet, M'^\bullet)).$$

In particular if $M^\bullet = M, M'^\bullet = M'$ are concentrated in degree 0, $H^n(\Gamma(M, M')) = 0$ for $n \geq 2$ and $n < 0$.

Proof. By the octahedron axiom we have the following triangle in $D^b(\text{Ab})$

$$\text{Ker } \psi_{M^\bullet, M'^\bullet} \rightarrow \Gamma(M^\bullet, M'^\bullet) \rightarrow \text{Coker } \psi_{M^\bullet, M'^\bullet}[-1] \xrightarrow{+}.$$

Its cohomological long exact sequence is

$$\xrightarrow{\partial} H^n(\text{Ker } \psi_{M^\bullet, M'^\bullet}) \rightarrow H^n(\Gamma(M^\bullet, M'^\bullet)) \rightarrow H^n(\text{Coker } \psi_{M^\bullet, M'^\bullet}[-1]) \xrightarrow{\partial}$$

Note that by construction $H^n(\text{Ker } \psi_{M^\bullet, M'^\bullet}) = \text{Hom}_{pHK}(M^\bullet, M'^\bullet[n])$. Let I be the family of quasi-isomorphisms $g : M'^\bullet \rightarrow M''^\bullet$, then $\text{Hom}_{pHD}(M^\bullet, M'^\bullet[n]) = \text{colim}_I \text{Hom}_{pHK}(M^\bullet, M''^\bullet[n])$. Thus the result is proved if we show that:

- i) $H^n(\Gamma(M^\bullet, M'^\bullet)) \cong H^n(\Gamma(M^\bullet, M''^\bullet))$ for any $g : M'^\bullet \rightarrow M''^\bullet$ quasi-isomorphism.
- ii) $\text{colim}_I H^n(\text{Coker } \psi_{M^\bullet, M''^\bullet}[-1]) = 0$.

The first claim follows from the exactness of $\Gamma(M^\bullet, -)$ and the second is proved in [Bei86, 1.7, 1.8] (and with more details in [Hub95, Lemma 4.2.8] or [Ban02, Lemma 2.15]) with the assumption that all the gluing maps are quasi-isomorphisms, but this hypothesis is not necessary. \square

Lemma 2.11 (Tensor Product). *Let $M^\bullet, M'^\bullet, I^\bullet$ be p -adic Hodge complexes. For any $\alpha \in K$ there is a morphism of complexes*

$$\cup_\alpha : \Gamma(I^\bullet, M^\bullet) \otimes \Gamma(I^\bullet, M'^\bullet) \rightarrow \Gamma(I^\bullet, M^\bullet \otimes M'^\bullet)$$

such that they are all equivalent up to homotopy.

Proof. See [Bei86, 1.11]. \square

Remark 2.12 (Enlarging the diagram). We recall some results from [Lev98, Ch. V, 2.3.3]. Let $M_1^\bullet \xrightarrow{f} M_2^\bullet \xleftarrow{g} M_3^\bullet$ (resp. $M_1^\bullet \xleftarrow{f} M_2^\bullet \xrightarrow{g} M_3^\bullet$) be a diagram of complexes in $C^b(K)$. Let $P^\bullet = \text{Cone}(f - g : M_1^\bullet \times M_3^\bullet \rightarrow M_2^\bullet)[-1]$ be the *quasi pull-back complex* (resp. $Q^\bullet = \text{Cone}((f, -g) : M_2^\bullet \rightarrow M_1^\bullet \times M_3^\bullet)$ be the *quasi push-out*). Assume that f is quasi-isomorphism then we have the following diagrams commutative up to homotopy

$$\begin{array}{ccc} P^\bullet & \xrightarrow{h} & M_3^\bullet \\ \downarrow & & \downarrow g \\ M_1^\bullet & \xrightarrow{f} & M_2^\bullet \end{array} \qquad \begin{array}{ccc} M_2^\bullet & \xrightarrow{f} & M_1^\bullet \\ g \downarrow & & \downarrow \\ M_3^\bullet & \xrightarrow{k} & P^\bullet \end{array}$$

such that h and k are quasi-isomorphisms.

Now let pHC' be a category of systems $(M_{\text{rig}}^\bullet, M_{\text{dR}}^\bullet, M_1^\bullet, M_2^\bullet, M_3^\bullet, c, s, f, g)$ similarly to Def. 2.5 and such that there is a diagram

$$M_{\text{rig}}^\bullet \xrightarrow{c} M_1^\bullet \xleftarrow{f} M_2^\bullet \xrightarrow{g} M_3^\bullet \xleftarrow{s} M_{\text{dR}}^\bullet$$

Then the quasi push-out induces a functor from the category pHC' to the category pHC . This functor is compatible with tensor product after passing to the categories pHK' and pHK .

3 Godement resolution

Here we recall some facts about the *generalized* Godement resolution, also called bar-resolution (we refer to [Ivo05], see also [Wei94, §8.6]).

Let $u : P \rightarrow X$ be a morphism of Grothendieck topologies so that P^\sim (resp. X^\sim) is the category of abelian sheaves on P (resp. X). Then we have a pair of adjoint functors (u^*, u_*) , where $u^* : X^\sim \rightarrow P^\sim$, $u_* : P^\sim \rightarrow X^\sim$. For any object \mathcal{F} of X^\sim we can define a co-simplicial object $B^*(\mathcal{F}) : \Delta \rightarrow X^\sim$ in the following way. First let $\eta : \text{id}_{X^\sim} \rightarrow u_* u^*$ and $\epsilon : u^* u_* \rightarrow \text{id}_{P^\sim}$ be the natural transformations induced by adjunction.

Endow $B^{n+1}(\mathcal{F}) := (u_* u^*)^n(\mathcal{F})$ with co-degeneracy maps

$$\sigma_i^n := (u_* u^*)^i u_* \epsilon u^* (u_* u^*)^{n-1-i} : B^{n+1}(\mathcal{F}) \rightarrow B^n(\mathcal{F}) \quad i = 0, \dots, n-1$$

and co-faces

$$\delta_i^{n-1} := (u_* u^*)^i \eta (u_* u^*)^{n-i} : B^n(\mathcal{F}) \rightarrow B^{n+1}(\mathcal{F}) \quad i = 0, \dots, n.$$

Lemma 3.1. *With the above notation let $sB^*(\mathcal{F})$ be the associated complex of objects of X^\sim . Then there is a canonical map $b_{\mathcal{F}} : \mathcal{F} \rightarrow sB^*(\mathcal{F})$ such that $u^*(b_{\mathcal{F}})$ is a quasi-isomorphism. Moreover if u^* is exact and conservative $b_{\mathcal{F}}$ is a quasi-isomorphism.*

Proof. See [Ivo05, Ch. III, Lemma 3.4.1]. \square

Then for any sheaf $\mathcal{F} \in X^\sim$ (or complex of sheaves) we can define a functorial map $b_{\mathcal{F}} : \mathcal{F} \rightarrow sB^*(\mathcal{F})$ with $sB^n(\mathcal{F}) := (u_* u^*)^{n+1} \mathcal{F}$. We will denote this complex of sheaves by $\text{Gd}_P(\mathcal{F})$. In the case u^* is exact and conservative $\text{Gd}_P(\mathcal{F})$ is a canonical resolution of \mathcal{F} . If \mathcal{F}^\bullet is a complex of sheaves on X we denote by $\text{Gd}_P(\mathcal{F}^\bullet)$ the simple complex $s(\text{Gd}_P(\mathcal{F}^i)^j)$. Often we will need to iterate this process and we will write $\text{Gd}_P^2(\mathcal{F}) := \text{Gd}_P(\text{Gd}_P(\mathcal{F}))$.

Now suppose there is a commutative diagram of sites

$$\begin{array}{ccc} P & \xrightarrow{g} & Q \\ u \downarrow & & \downarrow v \\ X & \xrightarrow{f} & Y \end{array}$$

and a morphism of sheaves $a : \mathcal{G} \rightarrow f_* \mathcal{F}$ where \mathcal{F} (resp. \mathcal{G}) is a sheaf on X (resp. Y).

Lemma 3.2. *There is a canonical map $\text{Gd}_Q(\mathcal{G}) \rightarrow f_* \text{Gd}_P(\mathcal{F})$ compatible with $b_{\mathcal{F}}$ and $b_{\mathcal{G}}$.*

Proof. We need only show that there is a canonical map $v_* v^* \mathcal{G} \rightarrow f_* u_* u^* \mathcal{F}$ lifting a . First consider the composition $\mathcal{G} \rightarrow f_* \mathcal{F} \rightarrow f_* u_* u^* \mathcal{F}$. Then we get a map $\mathcal{G} \rightarrow v_* g_* u^* \mathcal{F}$ because $v_* g_* = f_* u_*$. By adjunction this gives $v^* \mathcal{G} \rightarrow g_* u^* \mathcal{F}$. Then we apply v_* and use $v_* g_* = f_* u_*$ to obtain the desired map. \square

Remark 3.3 (Tensor product). The Godement resolution is compatible with tensor product, *i.e.*, for any pair of sheaves \mathcal{F}, \mathcal{G} on X there is a canonical quasi isomorphism $\text{Gd}_P(\mathcal{F}) \otimes \text{Gd}_P(\mathcal{G}) \rightarrow \text{Gd}_P(\mathcal{F} \otimes \mathcal{G})$. The same holds for complexes that are bounded below. (See [FS02, Appendix A])

Definition 3.4 (cf. [SGA4.1, Exp. IV, §6]). Let X be a site and $Sh(X)$ be the associated topos of sheaves of sets. A *point* of X is a morphism of topoi $\pi : Set \rightarrow Sh(X)$, *i.e.*, a pair of adjoint functors (π^*, π_*) such that π^* is left exact.

Example 3.5. Let X be a scheme. Then any point x of the topological space underlying X gives a point π_x of the Zariski site of X . We call them Zariski points.

Let now x be a geometric point of X , then it induces a point π_x for the étale site of X . We call them étale points of X .

Let \mathcal{F} be a Zariski (resp. étale) sheaf on X and P be the set of Zariski (resp. étale) points of X . Then the functor $\mathcal{F} \mapsto \sqcup_{\pi \in P} \mathcal{F}_\pi := \pi^* \mathcal{F}$ is exact and conservative. In other words the Zariski (resp. étale) site of X has enough points.

Definition 3.6 (Points on rigid analytic spaces [vdPS95]). Let X be a rigid analytic space over K . We recall that a *filter* f on X is a collection $(\mathcal{U}_\alpha)_\alpha$ of admissible open subsets of X satisfying:

- (i) $X \in f$, $\emptyset \notin f$;
- (ii) if $\mathcal{U}_\alpha, \mathcal{U}_\beta \in f$ then $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \in f$;
- (iii) if $\mathcal{U}_\alpha \in f$ and the admissible open \mathcal{V} contains \mathcal{U}_α , then $\mathcal{V} \in f$.

A *prime filter* on X is a filter p satisfying moreover

- (iv) if $\mathcal{U} \in p$ and $\mathcal{U} = \cup_{i \in I} \mathcal{U}'_i$ is an admissible covering of \mathcal{U} , then $\mathcal{U}'_{i_0} \in p$ for some $i_0 \in I$.

Let $P(X)$ be the set of all prime filters of X . The filters on X are ordered with respect to inclusion. We can give to $P(X)$ a Grothendieck topology and define a morphism of sites $\sigma : P(X) \rightarrow X$. Also we denote by $Pt(X)$ the set of prime filters with the discrete topology. Let $i : Pt(X) \rightarrow P(X)$ be the canonical inclusion and $\xi = \sigma \circ i$.

Remark 3.7. Let $p = (\mathcal{U}_\alpha)_\alpha$ be a prime filter on X as above. Then p is a point of the site X (see Def. 3.4). Using the construction of the continuous map σ of [vdPS95] we get that the morphism of topoi $\pi : Set \rightarrow Sh(X)$, associated to p , is defined in the following way: for any sheaf (of sets) \mathcal{F} on X let $\pi^*(\mathcal{F}) = \text{colim}_\alpha \mathcal{F}(\mathcal{U}_\alpha)$; for any set \mathcal{S} and \mathcal{V} admissible open in X , let

$$\pi_* \mathcal{S}(\mathcal{V}) = \begin{cases} \mathcal{S} & \text{if } \mathcal{V} = \mathcal{U}_\alpha \text{ for some } \alpha \\ 0 & \text{otherwise} \end{cases}$$

where 0 denotes the final object in the category Set . In fact with the above notations we get easily the adjunction

$$\text{Hom}_{Set}(\pi^*(\mathcal{F}), \mathcal{S}) = \lim_{\alpha} \text{Hom}_{Set}(\mathcal{F}(\mathcal{U}_\alpha), \mathcal{S}) = \text{Hom}_{Sh(X)}(\mathcal{F}, \pi_* \mathcal{S}).$$

Lemma 3.8. *With the above notation the functor $\xi^{-1} : Sh(X) \rightarrow Sh(Pt(X))$ is exact and conservative. In other words for any $p \in Pt(X)$ the functors $Sh(X) \ni \mathcal{F} \mapsto \mathcal{F}_p$ are exact and $\mathcal{F} = 0$ if $\mathcal{F}_p = 0$.*

Proof. See [vdPS95, §4] after the proof of the Theorem 1. □

4 Rigid and de Rham complexes

We begin this section by recalling the construction of the rigid complexes of [Bes00, §4]. Instead of the techniques of [SGA4.2, Vbis] we use the machinery of generalized Godement resolution as developed in Section 3. This alternative approach was also mentioned by Besser in the introduction of his paper.

We then recall the construction of the de Rham complexes.

We call a *rigid triple* a system (X, \bar{X}, P) where: X is an algebraic k -scheme; $j : X \rightarrow \bar{X}$ is an open embedding into a proper k -scheme; $\bar{X} \rightarrow P$ is a closed embedding into a p -adic formal \mathcal{V} -scheme P which is smooth in a neighborhood of X .

Definition 4.1 ([Bes00, 4.2, 4.4]). Let (X, \bar{X}, P) , (Y, \bar{Y}, Q) be two rigid triples and let $f : X \rightarrow Y$ be a morphism of k -schemes. Let $\mathcal{U} \subset]\bar{X}[P$ be a strict neighborhood of $]X[_P$ and $F : \mathcal{U} \rightarrow Q_K$ be a morphism of K -rigid spaces. We say that F is *compatible* with f if it induces the following commutative diagram

$$\begin{array}{ccc}]X[_P & \xrightarrow{F} &]Y[_Q \\ \downarrow sp & & \downarrow sp \\ X & \xrightarrow{f} & Y \end{array}$$

We write $\text{Hom}_f(\mathcal{U}, Q_K)$ for the collection of morphism compatible with f .

The collection of rigid triples forms a category with the following morphisms: $\text{Hom}((X, \bar{X}, P), (Y, \bar{Y}, Q)) =$ the set of pairs (f, F) where $f : X \rightarrow Y$ is a k -morphism and $F \in \text{colim}_{\mathcal{U}} \text{Hom}_f(\mathcal{U}, Q)$. We denote it by RT .

Proposition 4.2. (i) *There is a functor from the category of algebraic k -schemes Sch/k*

$$(\text{Sch}/k)^\circ \rightarrow C(K_0) \quad X \mapsto R\Gamma_{\text{rig}}(X/K_0)$$

such that $H^i(R\Gamma_{\text{rig}}(X/K_0)) \cong H_{\text{rig}}^i(X/K_0)$. Moreover there exists a canonical σ -linear endomorphism of $R\Gamma_{\text{rig}}(X/K_0)$ inducing the Frobenius on cohomology.

(ii) *There are two functors $\text{RT} \rightarrow C(K)$*

$$\widetilde{R\Gamma}_{\text{rig}}(X)_{\bar{X}, P}, \quad R\Gamma_{\text{rig}}(X/K)_{\bar{X}, P}$$

and functorial quasi-isomorphisms with respect to maps of rigid triples

$$R\Gamma_{\text{rig}}(X/K) \leftarrow \widetilde{R\Gamma}_{\text{rig}}(X)_{\bar{X}, P} \rightarrow R\Gamma_{\text{rig}}(X)_{\bar{X}, P}.$$

Proof. See [Bes00, 4.9, 4.21, 4.22]. □

Remark 4.3. The building block of the construction is the functor $R\Gamma_{\text{rig}}(X/K)_{\bar{X}, P}$. That complex is constructed with a system of compatible resolution of the over-convergent de Rham complexes $j_X^\dagger \Omega_{\mathcal{U}}^\bullet$ where \mathcal{U} runs over all strict neighborhood of the tube of X . Using Godement resolution we can explicitly define

$$R\Gamma_{\text{rig}}(X/K)_{\bar{X}, P} := \text{colim}_{\mathcal{U}} \Gamma(\mathcal{U}, \text{Gd}_{\text{an}} j_X^\dagger \text{Gd}_{\text{an}} \Omega_{\mathcal{U}}^\bullet)$$

where $\text{Gd}_{\text{an}} = \text{Gd}_{P_t(\mathcal{U})}$. This will be an essential ingredient for achieving the main results of the paper.

All the proofs of [Bes00, §4] work using this *Godement* complex. We recall that

$$R\Gamma_{\text{rig}}(X/K) := \text{colim}_{A \in \text{SET}_X^0} R\Gamma_{\text{rig}}(X/K)_{\bar{X}_A, P_A} \quad \widetilde{R\Gamma}_{\text{rig}}(X/K) := \text{colim}_{A \in \text{SET}_{(X, \bar{X}, P)}^0} R\Gamma_{\text{rig}}(X/K)_{\bar{X}_A, P_A}$$

where SET_X^0 and $\text{SET}_{(X, \bar{X}, P)}^0$ are filtered category of indexes.

With some modifications we can provide a compact support version of the above functors. We just need to be careful in the choice of morphisms of rigid triples.

Definition 4.4. Let $(X, \bar{X}, P), (Y, \bar{Y}, Q)$ be two rigid triples and let $f : X \rightarrow Y$ be a morphism of k -schemes. Let $F : \mathcal{U} \rightarrow Q_K$ be compatible with f (as in Def. 4.1). We say that F is *strict* if there is a commutative diagram

$$\begin{array}{ccccc}]X[_P & \longrightarrow & \mathcal{U} & \longleftarrow & \mathcal{U} \setminus]X[_P \\ \downarrow F & & \downarrow F & & \downarrow F \\]Y[_Q & \longrightarrow & \mathcal{V} & \longleftarrow & \mathcal{V} \setminus]Y[_Q \end{array}$$

where \mathcal{V} is a strict neighborhood of $]Y[_Q$ in \bar{Y}_Q

It is easy to show that strict morphisms are composable. We denote by RT_c the category of *rigid triples with proper morphisms* which is the (not full) sub-category of RT with the same objects and morphisms pairs (f, F) , where f is proper and F is a germ of a strict compatible morphism.

Lemma 4.5. (i) *Let (X, \bar{X}, P) be a rigid triple and let \mathcal{U} be a strict neighborhood of $]X[_P$. Then*

$$H^i(\Gamma(\mathcal{U}, \text{Gd}_{\text{an}} \Gamma_{]X[_P} \text{Gd}_{\text{an}} \Omega_{\mathcal{U}}^\bullet)) = H_{\text{rig}, c}^i(X).$$

(ii) *Let (Y, \bar{Y}, Q) be another rigid triple, $f : X \rightarrow Y$ be a proper k -morphism and let $F : \mathcal{U} \rightarrow Q_K$ be a morphism of K -analytic space compatible with f and strict. Then there is a canonical map*

$$F^* : \text{Gd}_{\text{an}} \Gamma_{]Y[_Q} \text{Gd}_{\text{an}} \Omega_{Q_K}^\bullet \rightarrow F_* \text{Gd}_{\text{an}} \Gamma_{]X[_P} \text{Gd}_{\text{an}} \Omega_{\mathcal{U}}^\bullet.$$

Proof. i) It is sufficient to note that $\mathrm{Gd}_{\mathrm{an}}(\Omega_{\mathcal{U}}^\bullet)$ is a complex of flasque sheaves and that a flasque sheaf is acyclic for $\Gamma_{|X|_{\mathbb{P}}}$. Let F be a flasque sheaf on the rigid analytic space \mathcal{U} . By definition $\Gamma_{|X|_{\mathbb{P}}} = \mathrm{Ker}(a : F \rightarrow i_* i^* F)$. It is easy to check that $\mathbb{R}^q i_* i^* F = 0$ for $q \geq 1$. Hence $\mathbb{R}\Gamma_{|X|_{\mathbb{P}}} F \cong \mathrm{Cone}(a : F \rightarrow i_* i^* F)[-1]$. But by hypothesis the map a is surjective so that $\mathrm{Cone}(a : F \rightarrow i_* i^* F)[-1] \cong \mathrm{Ker}(a) = \mathbb{R}^0 \Gamma_{|X|_{\mathbb{P}}} F$.

ii) First consider the canonical pull-back of differential forms $F^* : \Omega_{\mathbb{Q}_K} \rightarrow F_* \Omega_{\mathcal{U}}$. Then by Lemma 3.2 we get a map $\mathrm{Gd}_{P_t(\mathbb{Q}_K)} \Omega_{\mathbb{Q}_K}^\bullet \rightarrow F_* \mathrm{Gd}_{P_t(\mathcal{U})} \Omega_{\mathcal{U}}$. Applying the functor $\Gamma_{|X|_{\mathbb{P}}}$ to the adjoint map we get $\Gamma_{|X|_{\mathbb{P}}} F^{-1} \mathrm{Gd}_{P_t(\mathbb{Q}_K)} \Omega_{\mathbb{Q}_K} \rightarrow \Gamma_{|X|_{\mathbb{P}}} \mathrm{Gd}_{P_t(\mathcal{U})} \Omega_{\mathcal{U}}$. But the strictness of F implies that there is a canonical map $F^{-1} \Gamma_{|Y|_{\mathbb{Q}}} \rightarrow \Gamma_{|X|_{\mathbb{P}}} F^{-1}$ (see [LS07, Proof of Prop. 5.2.17]). Hence we have a map $F^{-1} \Gamma_{|Y|_{\mathbb{Q}}} \mathrm{Gd}_{P_t(\mathbb{Q}_K)} \Omega_{\mathbb{Q}_K} \rightarrow \Gamma_{|X|_{\mathbb{P}}} \mathrm{Gd}_{P_t(\mathcal{U})} \Omega_{\mathcal{U}}$. We can conclude the proof by taking the adjoint of this map and again applying Lemma 3.2. \square

Proposition 4.6. (i) *There is a functor from the category of algebraic k -schemes with proper morphisms Sch_c/k*

$$(\mathrm{Sch}_c/k)^\circ \rightarrow C(K_0) \quad X \mapsto R\Gamma_{\mathrm{rig},c}(X/K_0)$$

such that $H^i(R\Gamma_{\mathrm{rig},c}(X/K_0)) \cong H_{\mathrm{rig},c}^i(X/K_0)$. Moreover there exists a canonical σ -linear endomorphism of $R\Gamma_{\mathrm{rig},c}(X/K_0)$ inducing the Frobenius on cohomology.

(ii) *There are two functors $\mathrm{RT}_c \rightarrow C(K)$*

$$\widetilde{R\Gamma}_{\mathrm{rig},c}(X)_{\bar{X},P}, \quad R\Gamma_{\mathrm{rig},c}(X/K)_{\bar{X},P}$$

and functorial quasi-isomorphisms with respect to maps of rigid triples

$$R\Gamma_{\mathrm{rig},c}(X/K) \leftarrow \widetilde{R\Gamma}_{\mathrm{rig},c}(X)_{\bar{X},P} \rightarrow R\Gamma_{\mathrm{rig},c}(X)_{\bar{X},P}.$$

Proof. In view of Lemma 4.5 it is sufficient to mimic the construction given in [Bes00, 4.9, 4.21, 4.22] but using only proper morphisms of k -schemes and strict compatible maps. In this case the functors used in the construction are

$$R\Gamma_{\mathrm{rig},c}(X/K)_{\bar{X},P} := \mathrm{colim}_{\mathcal{U}} \Gamma(\mathcal{U}, \mathrm{Gd}_{\mathrm{an}} \Gamma_{|X|_{\mathbb{P}}} \mathrm{Gd}_{\mathrm{an}} \Omega_{\mathcal{U}}^\bullet)$$

and

$$R\Gamma_{\mathrm{rig},c}(X/K) := \mathrm{colim}_{A \in \mathrm{SET}_X^0} R\Gamma_{\mathrm{rig},c}(X/K)_{\bar{X}_A, P_A} \quad \widetilde{R\Gamma}_{\mathrm{rig},c}(X/K) := \mathrm{colim}_{A \in \mathrm{SET}_{(X,\bar{X},P)}^0} R\Gamma_{\mathrm{rig},c}(X/K)_{\bar{X}_A, P_A}.$$

\square

Now we focus on de Rham complexes and we deal with smooth K -algebraic schemes. Let X be a smooth algebraic K -scheme. The (algebraic) de Rham cohomology of X is the hyper-cohomology of its complex of Kähler differentials $H_{\mathrm{dR}}^i(X/K) := H^i(X, \Omega_{X/K}^\bullet)$ (See [Gro66b]). We can also define the de Rham cohomology with compact support (see [BCF04, §1]) as the hyper-cohomology groups $H_{\mathrm{dR},c}^i(X/K) := H^i(\bar{X}, \lim_n J^n \Omega_{\bar{X}/K}^\bullet)$ where $X \rightarrow \bar{X}$ is a smooth compactification and J is the sheaf of ideals associated to the complement $\bar{X} \setminus X$ (this definition does not depend on the choice of \bar{X} [BCF04, Thr. 1.8]). In order to consider the Hodge filtration on the de Rham cohomology groups we fix a normal crossings compactification $g : X \rightarrow Y$ and let $D := Y \setminus X$ be the complement divisor (This is possible by the Nagata Compactification Theorem and the Hironaka Resolution Theorem, see [Del71, §3.2.1]). We denote by $\Omega_Y^\bullet \langle D \rangle$ the de Rham complex of Y with logarithmic poles along D (in the Zariski topology) (See [Jan90, 3.3]). Let $I \subset \mathcal{O}_Y$ be the defining sheaf of ideals of D .

Proposition 4.7. *With the above notations*

(i) *there is a canonical isomorphism*

$$H_{\mathrm{dR}}^i(X/K) \cong H^i(Y, \Omega_Y^\bullet \langle D \rangle) \quad (\text{resp. } H_{\mathrm{dR},c}^i(X) \cong H^i(Y, I\Omega_Y^\bullet \langle D \rangle)).$$

(ii) *The spectral sequence*

$$E_1^{p,q} = H^q(Y, \Omega^p) \Rightarrow H^i(Y, \Omega^\bullet)$$

degenerates at 1 for $\Omega^\bullet = \Omega_Y^\bullet \langle D \rangle, I\Omega_Y^\bullet \langle D \rangle$.

(iii) The filtration induced by the above spectral sequence on $H_{\mathrm{dR}}^i(X)$ (resp. $H_{\mathrm{dR},c}^i(X)$) is independent by the choice of Y . Namely

$$F^j H_{\mathrm{dR}}^i(X) := H^i(Y, \sigma^{\geq j} \Omega_Y^\bullet \langle D \rangle) \quad (\text{resp. } H_{\mathrm{dR},c}^i(X) := H^i(Y, \sigma^{\geq j} I\Omega_Y^\bullet \langle D \rangle))$$

where $\sigma^{\geq j}$ is the stupid filtration.

Proof. Using the argument of [Del71, 3.2.11] we get the independence of the choice of Y . The same holds for $H_{\mathrm{dR},c}^i(X)$.

Since our base field K is of characteristic 0 we can find an embedding $\tau : K \rightarrow \mathbb{C}$. Then by GAGA we get an isomorphism of filtered vector spaces $H^i(Y, \Omega_Y^\bullet \langle D \rangle) \otimes_K \mathbb{C} \cong H^i(Y_h, \Omega_{Y_h}^\bullet \langle D_h \rangle)$ (resp. $H^i(Y, I\Omega_Y^\bullet \langle D \rangle) \otimes_K \mathbb{C} \cong H^i(Y_h, I_h \Omega_{Y_h}^\bullet \langle D_h \rangle)$), where $(-)_h$ is the complex analytification functor and I_h is the defining sheaf of D_h . Thus we conclude by [Del71, §3] (resp. [PS08, Part II, Ex. 7.25] for the compact support case). \square

Remark 4.8. The degeneracy of the spectral sequence in (ii) of the above proposition can be proved algebraically ([DI87]). We don't know an algebraic proof of the isomorphism in (i).

In the sequel a morphism of pairs (X, Y) as above is a commutative square

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ u \downarrow & & \downarrow v \\ X' & \xrightarrow{g'} & Y' \end{array}$$

We say that the morphism is strict if the square is cartesian.

The complex $\Omega_Y^\bullet \langle D \rangle$ (resp. $I\Omega_Y^\bullet \langle D \rangle$) is a complex of Zariski sheaves over Y functorial with respect to the pair (X, Y) (resp. strict morphisms of pairs). We can construct two different (generalized) Gode-ment resolutions (see § 3): one using Zariski points; the other via the K -analytic space associated to Y .

We will write $Pt(Y) = Pt(Y_{\mathrm{zar}})$ for the set of Zariski points of Y with the discrete topology; $Pt(Y_{\mathrm{an}})$ for the discrete site of rigid points (Def. 3.6) of Y_{an} ; $Pt(Y_{\mathrm{an}}) \sqcup Pt(Y)$ is the direct sum in the category of sites.

Proposition 4.9. *With the notations of Prop. 4.7 let $w : Y_{\mathrm{an}} \rightarrow Y_{\mathrm{zar}}$ be the canonical map from the rigid analytic site to the Zariski site of Y . Then for any Zariski sheaf Ω on Y there is a diagram*

$$\mathrm{Gd}_{Pt(Y)}(\Omega) \leftarrow \mathrm{Gd}_{Pt(Y_{\mathrm{an}}) \sqcup Pt(Y)}(\Omega) \rightarrow w_* \mathrm{Gd}_{Pt(Y_{\mathrm{an}})}(w^* \Omega)$$

If we further consider $\Omega = \Omega_Y^\bullet \langle D \rangle$ (resp. $\Omega = I\Omega_Y^\bullet \langle D \rangle$), then the diagram is functorial with respect to the pair (X, Y) (resp. (X, Y) and strict morphisms). The same holds true with Gd_Y^2 instead of Gd_Y .

Proof. The first claim follows from Lemma 3.2 applied to the following commutative diagram of sites

$$\begin{array}{ccccc} Pt(Y_{\mathrm{an}}) & \longrightarrow & Pt(Y_{\mathrm{an}}) \sqcup Pt(Y) & \longleftarrow & Pt(Y) \\ \downarrow & & \downarrow & & \downarrow \\ Y_{\mathrm{an}} & \xrightarrow{w} & Y_{\mathrm{zar}} & \xleftarrow{\mathrm{id}} & Y_{\mathrm{zar}} \end{array}$$

with respect to the canonical map $\Omega \rightarrow w_* w^* \Omega$.

The second claim follows from the functoriality of the complex $\Omega_Y^\bullet \langle D \rangle$ (resp. $I\Omega_Y^\bullet \langle D \rangle$). \square

Proposition 4.10. (i) *Let $g : X \rightarrow Y$ be a normal crossing compactification as in Prop. 4.7. Then there is a quasi isomorphism of complexes of sheaves*

$$\Omega_Y^\bullet \langle D \rangle \rightarrow \mathrm{Gd}_{Pt(Y)}^2(\Omega_Y^\bullet \langle D \rangle) \quad (\text{resp. } I\Omega_Y^\bullet \langle D \rangle \rightarrow \mathrm{Gd}_{Pt(Y)}^2 I\Omega_Y^\bullet \langle D \rangle)$$

and the stupid filtration on $\Omega_Y^\bullet \langle D \rangle$ (resp. $I\Omega_Y^\bullet \langle D \rangle$) induces a filtration on the right term of the morphism.

(ii) Let Sm/K (resp. Sm_c/K) be the category of algebraic and smooth K -schemes (resp. with proper morphisms). Let $D_{\mathrm{dR}}^b(K)$ the derived category of the exact category of filtered vector spaces. Then there exist two functors

$$R\Gamma_{\mathrm{dR}}(-) : (\mathrm{Sm}/K)^\circ \rightarrow D_{\mathrm{dR}}^b(K) \quad R\Gamma_{\mathrm{dR},c}(-) : (\mathrm{Sm}_c/K)^\circ \rightarrow D_{\mathrm{dR}}^b(K)$$

such that $R\Gamma_{\mathrm{dR}}(X) = \Gamma(Y, \mathrm{Gd}_{\mathrm{Pt}(Y)}^2(\Omega_Y^\bullet\langle D \rangle))$, $R\Gamma_{\mathrm{dR},c}(X) = \Gamma(Y, \mathrm{Gd}_{\mathrm{Pt}(Y)}^2 I\Omega_Y^\bullet\langle D \rangle)$, with the same notation of (i).

(iii) The filtered complexes $R\Gamma_{\mathrm{dR}}(X)$, $R\Gamma_{\mathrm{dR},c}(X)$ are strict (see 2.3).

Proof. i) This follows directly from the definition of Godement resolution.

ii) This follows from the functoriality of the Godement resolution with respect to morphism of pairs and [Del71, 3.2.11] or [Hub95, Lemma 15.2.3] for the compact support case.

iii) This follows by [PS08, Part II, §4.3, 7.3.1]. \square

5 Syntomic cohomology

In this section we construct the p -adic Hodge complexes needed to define the rigid syntomic cohomology groups (also with compact support) for a smooth algebraic \mathcal{V} -scheme. The functoriality will be a direct consequence of the construction.

Lemma 5.1. *Let $f : \mathcal{X} \rightarrow \mathcal{V}$ be a morphism of schemes and let $Z \subset \mathcal{X}_K$ be a closed sub-scheme of the generic fiber of \mathcal{X} . Then there exists a unique $\mathcal{Z} \subset \mathcal{X}$ closed sub-scheme which is flat over \mathcal{V} and which satisfies $\mathcal{Z}_K = Z$. Thus, \mathcal{Z} is the schematic closure of Z in \mathcal{X} .*

Proof. See [EGA4.3, 2.8.5]. \square

Proposition 5.2. *Let \mathcal{X} be a smooth scheme over \mathcal{V} . Then there exists a generic normal crossings compactification, i.e., an open embedding $g : \mathcal{X} \rightarrow \mathcal{Y}$ such that:*

- (i) \mathcal{Y} is proper over \mathcal{V} ;
- (ii) \mathcal{Y}_K is smooth over K ;
- (iii) $\mathcal{D}_K \subset \mathcal{Y}_K$ is a normal crossings divisor, where $\mathcal{D} = \mathcal{Y} \setminus \mathcal{X}$.

Proof. First by Nagata (see [Con07]) there exists an open embedding $\mathcal{X}_K \rightarrow Y$, where Y is a proper K -scheme. By the Hironaka resolution theorem we can assume that Y is a smooth compactification of \mathcal{X}_K with complement a normal crossings divisor. Now we can define \mathcal{Y}' to be the glueing of \mathcal{X} and Y along the common open sub-scheme \mathcal{X}_K . These schemes are all of finite type over \mathcal{V} . It follows from the construction that \mathcal{Y}' is a scheme, separated and of finite type over \mathcal{V} , whose generic fiber is Y .

The Nagata compactification theorem works also in a relative setting, namely for a separated and finite type morphism: hence we can find a \mathcal{V} -scheme \mathcal{Y} which is a compactification of \mathcal{Y}' over \mathcal{V} . Then we get that an open and dense embedding $h : \mathcal{Y}'_K = Y \rightarrow \mathcal{Y}_K$ of proper K -schemes. Hence h is the identity and the statement is proven. \square

Remark 5.3. We can also give another proof of the previous proposition assuming an embedding in a smooth \mathcal{V} -scheme. First by Nagata (see [Con07]) there exists an open embedding $\mathcal{X} \rightarrow \overline{\mathcal{X}}$. Now assume that $\overline{\mathcal{X}}$ is embeddable in a smooth \mathcal{V} -scheme \mathcal{W} then by [Wlo05, Th. 1.0.2] we can get a resolution of the K -scheme $\overline{\mathcal{X}}_K$ by making a sequence of blow-up with respect to a family of closed subset in good position with respect to the regular locus of $\overline{\mathcal{X}}_K$, in particular $Z_i \cap \mathcal{X}_K = \emptyset$. One can perform the same construction directly over \mathcal{V} , replacing the closed $Z_i \subset \mathcal{W}_K$ with their Zariski closure \mathcal{Z}_i in \mathcal{W} . By hypothesis $Z_i \subset \mathcal{W}_K \setminus \mathcal{X}_K$, hence its closure \mathcal{Z}_i is contained in $\mathcal{W} \setminus \mathcal{X}$ and $(\mathcal{Z}_i)_K = Z_i$, by Lemma 5.1. The construction doesn't affect what happens on the generic fiber because the blow-up construction is local and behaves well with respect to open immersions (See [Har77, Ch.II 7.15]). This will give \mathcal{Y} as in the proposition.

From now on we keep the notation of Prop. 5.2 with $g : \mathcal{X} \rightarrow \mathcal{Y}$ being fixed. To simplify the notations we denote by X (resp. \mathcal{Y}) the rigid analytic space associated to the generic fiber of \mathcal{X} (resp. \mathcal{Y}), usually denoted by $\mathcal{X}_K^{\text{an}}$. Let $w : \mathcal{Y} \rightarrow (\mathcal{Y}_K)_{\text{zar}}$ be the canonical map of sites. Sometimes we will simply write $\text{Gd}_{\text{an}} = \text{Gd}_{P_t(\mathcal{U})}$, if the rigid space \mathcal{U} is clear from the context. Similarly we write $\text{Gd}_{\text{zar}} = \text{Gd}_{P_t(X)}$ to denote the Godement resolution with respect to Zariski points of a K -scheme X .

Lemma 5.4. *With the above notations we have the following morphisms of complexes of K -vector spaces*

$$a : \Gamma(\mathcal{Y}, \text{Gd}_{\text{an}} j^\dagger \text{Gd}_{\text{an}} w^* \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle) \rightarrow \Gamma(X, \text{Gd}_{\text{an}} j^\dagger \text{Gd}_{\text{an}} \Omega_X^\bullet) \rightarrow R\Gamma_{\text{rig}}(\mathcal{X}_K/K)_{\mathcal{Y}_K, \widehat{\mathcal{Y}}}$$

and

$$b : \Gamma(\mathcal{Y}, \text{Gd}_{\text{an}} \Gamma_{\mathcal{X}_K} \text{Gd}_{\text{an}} w^* I\Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle) \rightarrow \Gamma(X, \text{Gd}_{\text{an}} \Gamma_{\mathcal{X}_K} \text{Gd}_{\text{an}} \Omega_X^\bullet) \rightarrow R\Gamma_{\text{rig}, c}(\mathcal{X}_K/K)_{\mathcal{Y}_K, \widehat{\mathcal{Y}}}.$$

All the maps are quasi-isomorphisms. We denote by a (resp. b) the composition of the maps in the first (resp. second) diagram

Proof. By construction (see Remark 4.3 and Prop. 4.6) $R\Gamma_{\text{rig}}(\mathcal{X}_K/K)_{\mathcal{Y}_K, \widehat{\mathcal{Y}}}$ (resp. $R\Gamma_{\text{rig}, c}(\mathcal{X}_K/K)_{\mathcal{Y}_K, \widehat{\mathcal{Y}}}$) is a direct limit of complexes indexed over the strict neighborhoods of $\mathcal{X}_K[\widehat{\mathcal{Y}}]$ and X is one of them. Hence the map on the right (of both diagrams) comes from the universal property of the direct limit.

For the map on the left consider first the canonical inclusion of algebraic differential forms with log poles into the analytic ones $w^* \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle \rightarrow g_*^{\text{an}} \Omega_X^\bullet$. By Lemma 3.2 we get a map $\text{Gd}_{P_t(\mathcal{Y})} w^* \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle \rightarrow g_*^{\text{an}} \text{Gd}_{P_t(X)} \Omega_X^\bullet$. Then applying the j^\dagger functor and noting that $j^\dagger g_*^{\text{an}} = g_*^{\text{an}} j^\dagger$ (see [LS07, 5.1.14]) one obtains a morphism $j^\dagger \text{Gd}_{P_t(\mathcal{Y})} w^* \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle \rightarrow g_*^{\text{an}} j^\dagger \text{Gd}_{P_t(X)} \Omega_X^\bullet$. This is what we need to apply Lemma 3.2 again and conclude the proof for the first diagram.

For the second diagram one repeats the same argument using [LS07, 5.2.15]. \square

Lemma 5.5. *With the above notations we have the following morphisms of complexes of K -vector spaces*

$$a' : \Gamma(\mathcal{Y}, \text{Gd}_{\text{an}}^2 w^* \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle) \rightarrow \Gamma(\mathcal{Y}, \text{Gd}_{\text{an}} j^\dagger \text{Gd}_{\text{an}} w^* \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle)$$

and

$$b' : \Gamma(\mathcal{Y}, \text{Gd}_{\text{an}} \Gamma_{\mathcal{X}_K} \text{Gd}_{\text{an}} w^* I\Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle) \rightarrow \Gamma(\mathcal{Y}, \text{Gd}_{\text{an}}^2 w^* I\Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle).$$

Proof. The map a' (resp. b') is induced by the canonical map $\Omega \rightarrow j^\dagger \Omega$ (resp. $\Gamma_{\mathcal{X}_K} \Omega \rightarrow \Omega$), where Ω is an abelian sheaf on \mathcal{Y} . In particular we consider $\Omega = \text{Gd}_{\text{an}} w^* \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle$ (resp. $\Omega = \text{Gd}_{\text{an}} w^* I\Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle$). To conclude the proof apply the functor Gd_{an} again and take global sections. \square

- 5.6 (Construction). Now we put together all we have done getting a diagram, say $R\Gamma'(\mathcal{X})$, of complexes of K -vector spaces

$$\begin{array}{ccccccc} R\Gamma_{\text{rig}}(\mathcal{X}/K) & & R\Gamma_{\text{rig}}(\mathcal{X}/K)_{\mathcal{Y}_K, \widehat{\mathcal{Y}}} & & \Gamma(\mathcal{Y}, \text{Gd}_{\text{an}}^2 w^* \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle) & & \Gamma(\mathcal{Y}_K, \text{Gd}_{\text{zar}}^2 \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle) \\ \alpha_1 \nearrow & & \nwarrow \alpha_2 & & \nearrow \alpha_3 & & \nwarrow \alpha_4 \\ R\Gamma_{\text{rig}}(\mathcal{X}/K_0) & & \widetilde{R\Gamma}_{\text{rig}}(\mathcal{X}/K)_{\mathcal{Y}_K, \widehat{\mathcal{Y}}} & & \Gamma(\mathcal{Y}, \text{Gd}_{\text{an}}^2 w^* \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle) & & \Gamma(\mathcal{Y}_K, \text{Gd}_{\text{an}+\text{zar}}^2 \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle) \\ & & & & \nearrow \alpha_5 & & \nwarrow \alpha_6 \\ & & & & \Gamma(\mathcal{Y}_K, \text{Gd}_{\text{zar}}^2 \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle) & & \Gamma(\mathcal{Y}_K, \text{Gd}_{\text{zar}}^2 \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle) \\ & & & & & & \nearrow \alpha_7 & \nwarrow \alpha_8 \end{array}$$

where $\alpha_1, \alpha_5, \alpha_8$ are the identity maps; α_2, α_3 are the maps of Prop. 4.2; α_4 is the composition of $a \circ a'$ (see Lemma 5.4 and Lemma 5.5); and α_6, α_7 are defined in Prop. 4.9. By repeatedly applying the quasi-pushout construction we obtain a diagram of the following shape

$$R\Gamma_{\text{rig}}(\mathcal{X}/K_0) \rightarrow R\Gamma_K(\mathcal{X}) \leftarrow R\Gamma_{\text{dR}}(\mathcal{X}). \quad (3)$$

It represents an object of pHC which we denote by $R\Gamma(\mathcal{X})$.

Similarly we can construct a p -adic Hodge complex $R\Gamma_c(\mathcal{X})$ associated to the diagram $R\Gamma'_c(\mathcal{X})$ defined as follows

$$\begin{array}{ccccccc}
R\Gamma_{\text{rig},c}(\mathcal{X}/K) & & R\Gamma_{\text{rig},c}(\mathcal{X}/K)_{\mathcal{Y}_k, \mathcal{Y}} & & \Gamma(\mathcal{Y}, \text{Gd}_{\text{an}}^2 w^* I\Omega_{\mathcal{Y}_K}(\mathcal{D}_K)) & & \Gamma(\mathcal{Y}_K, \text{Gd}_{\text{zar}}^2 I\Omega_{\mathcal{Y}_K}(\mathcal{D}_K)) \\
\beta_1 \nearrow & & \beta_2 \nwarrow & & \beta_3 \nearrow & & \beta_4 \nwarrow \\
R\Gamma_{\text{rig},c}(\mathcal{X}/K_0) & \xrightarrow{\sim} & \widetilde{R\Gamma}_{\text{rig},c}(\mathcal{X}/K)_{\mathcal{Y}_k, \mathcal{Y}} & \xrightarrow{\sim} & \Gamma(\mathcal{Y}, \text{Gd}_{\text{an}} \Gamma_{|X|} \text{Gd}_{\text{an}} w^* I\Omega_{Y_K}(D)) & \xrightarrow{\sim} & \Gamma(\mathcal{Y}_K, \text{Gd}_{\text{an}+\text{zar}}^2 I\Omega_{\mathcal{Y}_K}(\mathcal{D}_K)) \\
& & \beta_5 \nearrow & & \beta_6 \nwarrow & & \beta_7 \nearrow \\
& & \Gamma(\mathcal{Y}, \text{Gd}_{\text{an}} \Gamma_{|X|} \text{Gd}_{\text{an}} w^* I\Omega_{Y_K}(D)) & \xrightarrow{\sim} & \Gamma(\mathcal{Y}_K, \text{Gd}_{\text{an}+\text{zar}}^2 I\Omega_{\mathcal{Y}_K}(\mathcal{D}_K)) & \xrightarrow{\sim} & \Gamma(\mathcal{Y}_K, \text{Gd}_{\text{zar}}^2 I\Omega_{\mathcal{Y}_K}(\mathcal{D}_K)) \\
& & & & \beta_8 \nwarrow & &
\end{array}$$

where β_1, β_8 are the identity maps; β_2, β_3 are the maps of Prop. 4.2; β_4 is the map b of Lemma 5.4; $\beta_5 = b'$ of Lemma 5.5; and β_6, β_7 are defined in Prop. 4.9. Note that β_6 is a quasi-isomorphism by GAGA.

Proposition 5.7. *Let Sm/R (resp. Sm_c/R) be the category of algebraic and smooth R -schemes (resp. with proper morphisms). The previous construction induces the following functors*

$$R\Gamma(-) : (\text{Sm}/\mathcal{V})^\circ \rightarrow pHD \quad R\Gamma_c(-) : (\text{Sm}_c/\mathcal{V})^\circ \rightarrow pHD$$

Proof. Let $f : \mathcal{X} \rightarrow \mathcal{X}'$ be a morphism of smooth \mathcal{V} -schemes. To get the functoriality we just have to show that can find two gncd compactifications $g : \mathcal{X} \rightarrow \mathcal{Y}$ and $g' : \mathcal{X}' \rightarrow \mathcal{Y}'$ and a map $h : \mathcal{Y} \rightarrow \mathcal{Y}'$ extending f , i.e., $hg = g'f$. We argue as in [Del71, §3.2.11]. Fix two gncd compactifications $g' : \mathcal{X}' \rightarrow \mathcal{Y}'$ and $l : \mathcal{X} \rightarrow \mathcal{Z}$. Then consider the canonical map $\mathcal{X} \rightarrow \mathcal{Z} \times \mathcal{Y}'$ induced by l and $g'f$. Let $\tilde{\mathcal{X}}$ be the closure of \mathcal{X} in $\mathcal{Z} \times \mathcal{Y}'$ and use the same argument as in the proof of Prop. 5.2 to get \mathcal{Y} which is generically a resolution of the singularities of $\tilde{\mathcal{X}}$. Then by Prop. 4.9 and Prop. 4.2 we get the functoriality of $R\Gamma(-)$.

If we further assume f to be proper then we can apply the same argument in order to get the commutative square $hg = g'f$ as above. Then by properness this square is also cartesian by [Hub95, Lemma 15.2.3]. From this fact and Prop. 4.6, 4.9 we obtain the functoriality of $R\Gamma_c(-)$ with respect to proper maps. \square

Definition 5.8. Let \mathcal{X} be a smooth algebraic scheme over \mathcal{V} . For any n, i integers we define the *absolute cohomology groups* of \mathcal{X}

$$H_{\text{abs}}^n(\mathcal{X}, i) := \text{Hom}_{pHD}(\mathbb{K}, R\Gamma(\mathcal{X})(i)[n]) \quad (= H^n(\Gamma(\mathbb{K}, R\Gamma(\mathcal{X})(i)))); \quad (4)$$

the *absolute cohomology with compact support* groups of \mathcal{X}

$$H_{\text{abs},c}^n(\mathcal{X}, i) := \text{Hom}_{pHD}(\mathbb{K}, R\Gamma_c(\mathcal{X})(i)[n]) \quad (= H^n(\Gamma(\mathbb{K}, R\Gamma_c(\mathcal{X})(i)))) . \quad (5)$$

A direct consequence of the definition is the existence of the following long exact sequence which should be considered to be a p -adic analog of the corresponding sequence for Deligne-Beilinson cohomology [Bei84, Introduction].

Proposition 5.9. *With the above notation we have the following long exact sequences*

$$\cdots \rightarrow H_{\text{abs}}^n(\mathcal{X}, i) \rightarrow H_{\text{rig}}^n(\mathcal{X}_k/K_0) \times F^i H_{\text{dR}}^n(\mathcal{X}_K) \xrightarrow{h} H_{\text{rig}}^n(\mathcal{X}_k/K_0) \times H_{\text{rig}}^n(\mathcal{X}_k/K) \xrightarrow{+} \cdots$$

where $h(x_0, x_{\text{dR}}) = (\phi(x_0^\sigma) - p^i x_0, x_0 \otimes 1_K - sp(x_{\text{dR}}))$;

$$\cdots \rightarrow H_{\text{abs},c}^n(\mathcal{X}, i) \rightarrow H_{\text{rig},c}^n(\mathcal{X}_k/K_0) \times F^i H_{\text{dR},c}^n(\mathcal{X}_K) \xrightarrow{h_c} H_{\text{rig},c}^n(\mathcal{X}_k/K_0) \times H_{\text{dR},c}^n(\mathcal{X}_K/K) \xrightarrow{+} \cdots$$

where $h_c(x_0, x_{\text{dR}}) = (\phi_c(x_0\sigma) - p^i x_0, cosp(x_0 \otimes 1_K) - x_{\text{dR}})$;

Proof. By Prop. 2.10 the absolute cohomology is the cohomology of a mapping cone, namely $\Gamma(\mathbb{K}, R\Gamma(\mathcal{X}))$ (or $\Gamma_c(\mathbb{K}, R\Gamma(\mathcal{X}))$ for the compact support case). The above long exact sequences are easily induced from the distinguished triangle defined by the term of the mapping cone (see Remark. 2.9 (iii)). Indeed, one need only note that the map $c : R\Gamma_{\text{rig}}(\mathcal{X}/K_0) \otimes K \rightarrow R\Gamma_K(\mathcal{X})$ (resp. $s : R\Gamma_{\text{dR}}(\mathcal{X}) \rightarrow R\Gamma_K(\mathcal{X})$) is a quasi-isomorphism. \square

Proposition 5.10. *With the above notation there is a canonical isomorphism between the absolute cohomology we have defined and the (rigid) syntomic cohomology of Besser*

$$H_{\text{syn}}^n(\mathcal{X}, i) \cong H_{\text{abs}}^n(\mathcal{X}, i).$$

Proof. Besser defines a complex

$$\mathbb{R}\Gamma_{\text{Bes}}(\mathcal{X}, i) := \text{Cone}(\mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus \text{Fil}^i \mathbb{R}\Gamma_{\text{dR}}(\mathcal{X}) \rightarrow \mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus \mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}/K))[-1]$$

and the syntomic cohomology groups (of degree n and twisted i) by $H^n(\mathbb{R}\Gamma_{\text{Bes}}(\mathcal{X}, i))$ (cf. [Bes00, Proof of Prop. 6.3]). Note that, modulo the choice of the flasque resolution, $\mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}/K_0) = R\Gamma_{\text{rig}}(\mathcal{X}/K_0)$ (the left hand side is the notation used by Besser with the bold \mathbb{R}); the complex $\text{Fil}^i \mathbb{R}\Gamma_{\text{dR}}(\mathcal{X})$ is a direct limit over all the normal crossing compactifications of \mathcal{X}_K of complexes of the form $\Gamma(Y, \text{Gd}_{P(Y)}^2 \sigma^{\geq i} \Omega_Y(D))$, so that our $F^i R\Gamma_{\text{dR}}(\mathcal{X})$ is an element of this direct limit. To conclude the proof, recall that by Remark 2.9 (iii) we obtain

$$H_{\text{abs}}^n(\mathcal{X}, i) \cong H^n(\text{Cone}(R\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus F^i R\Gamma_{\text{dR}}(\mathcal{X}) \rightarrow R\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus R\Gamma_{\text{rig}}(\mathcal{X}/K))[-1])$$

and it is easy to check that the maps in the two mapping cones are defined in the same way. \square

Remark 5.11. The isomorphism $H^n(R\Gamma_{\text{syn}}(\mathcal{X}, i)) \cong \text{Hom}_{p\text{HD}}(\mathbb{K}, R\Gamma(\mathcal{X})(i)[n])$ can be viewed as a generalization of the result of Bannai [Ban02]. He considers only smooth schemes \mathcal{X} with a fixed compactification \mathcal{Y} and such that $\mathcal{Y} \setminus \mathcal{X} = \mathcal{D}$ is a relative normal crossings divisor over \mathcal{V} . Moreover, Bannai's construction is not functorial with respect to \mathcal{X} : functoriality holds only with respect to a so called syntomic datum. We should point out that the category defined by Bannai is endowed with a t-structure whose heart is the category MF_K^f of (weakly) admissible filtered Frobenius modules. In our case we don't have such a nice picture.

Remark 5.12. The category of p -adic Hodge complexes is not endowed with internal Hom. This is due to the fact that the Frobenius is only a quasi-isomorphism hence we cannot invert it. In particular this happens for the complex $R\Gamma_c(\mathcal{X})$: thus we cannot define the dual $R\Gamma_c(\mathcal{X})^\vee := \mathbb{R}\text{Hom}(R\Gamma_c(\mathcal{X}), \mathbb{K})$ and an absolute homology theory as

$$H_n^{\text{abs}}(\mathcal{X}, i) := \text{Hom}_{p\text{HD}}(\mathbb{K}, R\Gamma_c(\mathcal{X})^\vee(-i)[-n]) \quad (\text{cf. [Hub95, §15.3]}).$$

Nevertheless the usual adjunction between Hom and \otimes should give a natural isomorphism

$$\text{Hom}_{p\text{HD}}(\mathbb{K}, R\Gamma_c(\mathcal{X})^\vee) \cong \text{Hom}_{p\text{HD}}(R\Gamma_c(\mathcal{X}), \mathbb{K})$$

and the right term does makes sense in our setting. This motivates the following definition.

Definition 5.13. With the notation of Def. 5.8 we define the *absolute homology groups* of \mathcal{X} as

$$H_n^{\text{abs}}(\mathcal{X}, i) := \text{Hom}_{p\text{HD}}(R\Gamma_c(\mathcal{X}), \mathbb{K}(-i)[-n]) \quad (= H^{-n}(\Gamma(R\Gamma_c(\mathcal{X}), \mathbb{K}(-i)))) . \quad (6)$$

6 Cup product and Gysin map

We are going to prove that there is a morphism of p -adic Hodge complexes

$$R\Gamma(\mathcal{X}) \otimes R\Gamma_c(\mathcal{X}) \rightarrow R\Gamma_c(\mathcal{X})$$

This induces a pairing at the level of the complexes computing absolute cohomology. The key point is the compatibility of the de Rham and rigid pairings with respect to the specialization and co-specialization maps.

Let us start by fixing some notation. Let \mathcal{X} be a smooth algebraic \mathcal{V} -scheme; $g : \mathcal{X} \rightarrow \mathcal{Y}$ be a gncd compactification; $\mathcal{D} = \mathcal{Y} \setminus \mathcal{X}$ be the complement; \mathcal{Y} is the rigid analytic space associated to the K -scheme \mathcal{Y}_K ; as before we denote by $w : \mathcal{Y} \rightarrow \mathcal{Y}_{K, \text{zar}}$ the canonical morphism of sites (see notations after Prop. 5.2). For any $\mathcal{O}_{\mathcal{Y}_K}$ -module F we denote its pull-back by w^* .

Remark 6.1. (i) The wedge product of algebraic differentials induces the following pairing

$$p_{\mathrm{dR}} : \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle \otimes I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle \longrightarrow I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle .$$

(ii) The analytification of p_{dR} gives a pairing

$$w^* \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle \otimes w^* I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle \rightarrow w^* I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle$$

Hence by [Ber97a, Lemma 2.1] we get the following pairing

$$p_{\mathrm{rig}} : j^\dagger w^* \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle \otimes \Gamma_{\lfloor \mathcal{X}_k \rfloor} w^* I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle \rightarrow \Gamma_{\lfloor \mathcal{X}_k \rfloor} w^* I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle .$$

Lemma 6.2 (Sheaves level). *The following diagram commutes*

$$\begin{array}{ccc} \mathrm{Gd}_{\mathrm{an}}^2(w^* \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) \otimes \mathrm{Gd}_{\mathrm{an}}^2(w^* I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) & \xrightarrow{\quad} & \mathrm{Gd}_{\mathrm{an}}^2(w^* I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) \\ \uparrow & & \uparrow \\ \mathrm{Gd}_{\mathrm{an}}^2(w^* \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) \otimes \mathrm{Gd}_{\mathrm{an}} \Gamma_{\lfloor \mathcal{X}_k \rfloor} \mathrm{Gd}_{\mathrm{an}}(w^* I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) & \xrightarrow{m} & \mathrm{Gd}_{\mathrm{an}} \Gamma_{\lfloor \mathcal{X}_k \rfloor} \mathrm{Gd}_{\mathrm{an}}(w^* I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) \\ \downarrow & & \downarrow 1 \\ \mathrm{Gd}_{\mathrm{an}} j^\dagger \mathrm{Gd}_{\mathrm{an}}(w^* \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) \otimes \mathrm{Gd}_{\mathrm{an}} \Gamma_{\lfloor \mathcal{X}_k \rfloor} \mathrm{Gd}_{\mathrm{an}}(w^* I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) & \xrightarrow{\quad} & \mathrm{Gd}_{\mathrm{an}} \Gamma_{\lfloor \mathcal{X}_k \rfloor} \mathrm{Gd}_{\mathrm{an}}(w^* I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) \end{array}$$

where $m := p_{\mathrm{rig}} \circ (j^\dagger \otimes 1)$ by definition.

Proof. The bottom square commutes by construction. Also we get $p_{\mathrm{rig}} \circ (j^\dagger \otimes 1) = w^* p_{\mathrm{dR}}$ restricted to $w^* \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle \otimes \Gamma_{\lfloor \mathcal{X}_k \rfloor} w^* I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle$. \square

Proposition 6.3. *Let \mathcal{X} be a smooth \mathcal{V} -scheme. Then there exists a morphism of p -adic Hodge complexes*

$$\pi : R\Gamma(\mathcal{X}) \otimes R\Gamma_c(\mathcal{X}) \rightarrow R\Gamma_c(\mathcal{X})$$

which is functorial with respect to \mathcal{X} (as a morphism in $p\mathrm{HD}$). Moreover taking the cohomology of this map we get the following compatibility

$$\begin{array}{ccc} H_{\mathrm{dR}}^n(\mathcal{X}_K) \otimes H_{\mathrm{dR},c}^m(\mathcal{X}_K) & \longrightarrow & H_{\mathrm{dR},c}^{n+m}(\mathcal{X}_K) \\ \uparrow \mathrm{id} \otimes \mathrm{cosp} & & \uparrow \mathrm{cosp} \\ H_{\mathrm{dR}}^n(\mathcal{X}_K) \otimes H_{\mathrm{rig},c}^m(\mathcal{X}_K) & \longrightarrow & H_{\mathrm{rig},c}^{n+m}(\mathcal{X}_K) \\ \downarrow \mathrm{sp} \otimes \mathrm{id} & & \downarrow \mathrm{id} \\ H_{\mathrm{rig}}^n(\mathcal{X}_K) \otimes H_{\mathrm{rig},c}^m(\mathcal{X}_K) & \longrightarrow & H_{\mathrm{rig},c}^{n+m}(\mathcal{X}_K) \end{array}$$

Proof. It is sufficient to provide a pairing of the enlarged diagrams (see Remark 2.12). Thus we have to define a morphism of diagrams $\pi' : R\Gamma'(\mathcal{X}) \otimes R\Gamma'_c(\mathcal{X}) \rightarrow R\Gamma'_c(\mathcal{X})$ (notations as in § 5.6). Then it is easy to construct π' using the previous lemma and the compatibility of the Godement resolution with the tensor product. \square

Corollary 6.4. *There is a functorial pairing (induced by π of the previous proposition)*

$$H_{\mathrm{abs}}^n(\mathcal{X}, i) \otimes H_{\mathrm{abs},c}^m(\mathcal{X}, j) \rightarrow H_{\mathrm{abs},c}^{n+m}(\mathcal{X}, i+j) .$$

Proof. Consider the pairing of the Prop. 6.3. It induces a morphism $R\Gamma(\mathcal{X})(i) \otimes R\Gamma_c(\mathcal{X})(j) \rightarrow R\Gamma_c(\mathcal{X})(i+j)$. We then get the corollary by Def. 5.8 and Lemma 2.11. \square

Proposition 6.5 (Poincaré duality). *Let \mathcal{X} be a smooth and algebraic \mathcal{V} -scheme of dimension d . Then there is a canonical isomorphism*

$$H_{\mathrm{abs}}^n(\mathcal{X}, i) \cong H_{2d-n}^{\mathrm{abs}}(\mathcal{X}, d-i) .$$

Proof. By definition (see equations (4) and (6)) it is sufficient to prove that the complex $\Gamma(\mathbb{K}, R\Gamma(\mathcal{X}))$ is quasi-isomorphic to the diagram $\Gamma(R\Gamma_c(\mathcal{X}), \mathbb{K}[-2d](-d))$. First recall that $\Gamma(\mathbb{K}, R\Gamma(\mathcal{X}))$ is defined as

$$\text{Cone}(R\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus R\Gamma_K(\mathcal{X}) \oplus F^0 R\Gamma_{\text{dR}}(\mathcal{X}) \xrightarrow{\psi} R\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus R\Gamma_K(\mathcal{X}) \oplus R\Gamma_K(\mathcal{X}))[-1]$$

where $\psi(x_0, x_K, x_{\text{dR}}) = (\phi(x_0) - x_0, c(x_0 \otimes \text{id}_K) - x_K, x_K - s(x_{\text{dR}}))$. In order to define the desired map we need to modify this complex replacing $R\Gamma_K(\mathcal{X})$ with $R\Gamma_{\text{dR}}(\mathcal{X})$ as follows

$$\text{Cone}(R\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus R\Gamma_{\text{dR}}(\mathcal{X}) \oplus F^0 R\Gamma_{\text{dR}}(\mathcal{X}) \xrightarrow{\psi'} R\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus R\Gamma_K(\mathcal{X}) \oplus R\Gamma_{\text{dR}}(\mathcal{X}))[-1]$$

where $\psi'(x_0, x'_{\text{dR}}, x_{\text{dR}}) = (\phi(x_0) - x_0, c(x_0 \otimes \text{id}_K) - s(x'_{\text{dR}}), x'_{\text{dR}} - x_{\text{dR}})$. It is easy to see that this new complex, call it M^\bullet , is quasi-isomorphic to $\Gamma(\mathbb{K}, R\Gamma(\mathcal{X}))$.

Recall that the filtered complex $R\Gamma_{\text{dR},c}(\mathcal{X})$ is strict, so that the truncation $\tau_{\geq 2d} R\Gamma_{\text{dR},c}(\mathcal{X})$ is the usual truncation of complexes of K -vector spaces (see lemma 2.6 (iii) and remark 2.3). Then the cup product induces a morphism of complexes $M^\bullet \rightarrow \Gamma(R\Gamma_c(\mathcal{X}), \tau_{\geq 2d} R\Gamma_c(\mathcal{X}))$ which is a quasi-isomorphism by the Poincaré duality theorems for rigid and de Rham cohomology (see [Ber97a], [Hub95]). Explicitly this map is induced by the following commutative diagram

$$\begin{array}{ccc} R\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus R\Gamma_{\text{dR}}(\mathcal{X}) \oplus F^0 R\Gamma_{\text{dR}}(\mathcal{X}) & \xrightarrow{\alpha} & \text{Hom}_{K_0}^\bullet(N_0, \tau_{\geq 2d} N_0) \oplus \text{Hom}_K^\bullet(N_K, \tau_{\geq 2d} N_K) \oplus \text{Hom}_K^{\bullet,F}(N_{\text{dR}}, \tau_{\geq 2d} N_{\text{dR}}) \\ \downarrow \psi' & & \downarrow \xi \\ R\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus R\Gamma_{\text{rig}}(\mathcal{X}/K) \oplus R\Gamma_{\text{dR}}(\mathcal{X}) & \xrightarrow{\beta} & \text{Hom}_{K_0}^\bullet(N_0^\sigma, \tau_{\geq 2d} N_0) \oplus \text{Hom}_K^\bullet(N_{\text{rig}}, \tau_{\geq 2d} N_K) \oplus \text{Hom}_K^\bullet(N_{\text{dR}}, \tau_{\geq 2d} N_K) \end{array}$$

where $N := R\Gamma_c(\mathcal{X})$;

$$\xi(f_0, f_K, \text{dR}) = (\phi_c \circ f_0^\sigma - f_0 \circ \phi_c, c \circ (f_0 \otimes \text{id}_K) - f_K \circ c, f_K \circ s - s \circ x_{\text{dR}});$$

$$\alpha(x_0, x'_{\text{dR}}, x_{\text{dR}}) : (y_0, y_K, y_{\text{dR}}) \mapsto (x_0 \cup y_0, s(x'_{\text{dR}}) \cup y_K, x_{\text{dR}} \cup y_{\text{dR}});$$

$$\beta(x_0, x_{\text{rig}}, x_{\text{dR}}) : (y_0, y_{\text{rig}}, y_{\text{dR}}) \mapsto (x_0 \cup \phi_c(y_0), x_{\text{rig}} \cup y_{\text{rig}}, x_{\text{dR}} \cup y_{\text{dR}}).$$

To conclude the proof it is sufficient to apply the exact functor $\Gamma(R\Gamma_c(\mathcal{X}), -)$ to the following quasi-isomorphisms

$$\tau_{\geq 2d} R\Gamma_c(\mathcal{X}) \leftarrow H^{2d}(R\Gamma_c(\mathcal{X}))[-2d] \rightarrow \mathbb{K}(-d)[-2d].$$

□

Remark 6.6. We would like to point out some technical issues regarding Poincaré duality in syntomic cohomology.

- (i) If it were possible to define an internal Hom in the category pHC of p -adic Hodge complexes (see Remark 5.12), then by Prop. 6.3 one would obtain the following natural isomorphism

$$R\Gamma(\mathcal{X})(d)[2d] \cong R\Gamma_c(\mathcal{X})^\vee$$

in the triangulated category pHD , where $R\Gamma_c(\mathcal{X})^\vee := \mathbb{R}\text{Hom}_{pHC}(R\Gamma_c(\mathcal{X}), \mathbb{K})$ is the dual of $R\Gamma_c(\mathcal{X})$.

Then one would get the duality by adjunction

$$\text{Hom}_{pHD}(\mathbb{K}, R\Gamma(X)(i)[n]) \cong \text{Hom}_{pHD}(\mathbb{K}, R\Gamma_c(X)^\vee(i-d)[n-2d]) \cong \text{Hom}_{pHD}(R\Gamma_c(X)(d-i)[2d-n], \mathbb{K}).$$

- (ii) The Grothendieck-Leray spectral sequence for absolute homology is

$$E_2^{p,q} = \text{Hom}_{pHD}(H^{-q}(R\Gamma_c(\mathcal{X})(i))[-p], \mathbb{K}),$$

and it degenerates to the short exact sequences

$$0 \rightarrow \text{Hom}_{pHD}(H^n(R\Gamma_c(\mathcal{X})(i)), \mathbb{K}) \rightarrow H_n^{\text{syn}}(\mathcal{X}, i) \rightarrow \text{Hom}_{pHD}(H^{n+1}(R\Gamma_c(\mathcal{X})(i)), \mathbb{K}[1]) \rightarrow 0.$$

It follows directly from Prop. 2.10 that the group $\mathrm{Hom}_{pHD}(H^n(R\Gamma_c(\mathcal{X}))(i), \mathbb{K})$ is

$$\{(x_0, x_{dR}) \in H_{\mathrm{rig},c}^n(\mathcal{X}_k/K_0)^\vee \times (H_{\mathrm{dR},c}^n(\mathcal{X}_K)/F^{i+1})^\vee : (x_0 \otimes 1_K) = x_{dR} \circ \mathrm{cos}p, x_0 \circ \phi_c = p^i x_0^\sigma\}.$$

In cohomology the Frobenius is an invertible, hence the equation $x_0 \circ \phi_c = p^i x_0^\sigma$ is equivalent to $x_0 = p^i x_0^\sigma \phi_c^{-1}$ where the latter is the formula for the Frobenius of internal $\mathcal{H}om(H_{\mathrm{rig},c}^n(\mathcal{X}_k), K(-i))$ in the category of isocrystals (*i.e.*, modules with Frobenius). Hence by Poincaré duality for rigid cohomology we get $\phi((x_0^\vee)^\sigma) = p^{d-i} x_0^\vee$, $x^\vee \in H_{\mathrm{rig}}^{2d-n}(\mathcal{X}_k/K_0)$ is the cohomology class corresponding to x_0 , where ϕ is the Frobenius of $H_{\mathrm{rig}}^{2d-n}(\mathcal{X}_k/K_0)$.

- (iii) The absolute homology is defined via the complex $R\Gamma_c(\mathcal{X})$, but it is not clear how to relate it to the dual of the absolute cohomology with compact support.

Corollary 6.7 (Gysin map). *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a proper morphism of smooth algebraic \mathcal{V} -schemes of relative dimension d and e , respectively. Then there is a canonical map*

$$f_* : H_{\mathrm{abs}}^n(\mathcal{X}, i) \rightarrow H_{\mathrm{abs}}^{n+2c}(\mathcal{Y}, i+c)$$

where $c = e - d$.

Proof. This is a direct consequence of the previous proposition and the functoriality of $R\Gamma_c(\mathcal{X})$ with respect to proper morphisms. \square

Remark 6.8. With the above notations we get a morphism of spectral sequences

$$g : E_2^{p,q}(\mathcal{X}) := \mathrm{Hom}_{pHD}(\mathbb{K}, H^q(\mathcal{X})(i)[p]) \rightarrow E_2^{p,q+2c}(\mathcal{Y}) := \mathrm{Hom}_{pHD}(\mathbb{K}, H^{q+2c}(\mathcal{Y})(i)[p])$$

compatible with f_* . The map g is induced by the Gysin morphism in de Rham and rigid cohomology.

References

- [Ban02] Kenichi Bannai. Syntomic cohomology as a p -adic absolute Hodge cohomology. *Math. Z.*, 242(3):443–480, 2002.
- [BCF04] Francesco Baldassarri, Maurizio Cailotto, and Luisa Fiorot. Poincaré duality for algebraic de Rham cohomology. *Manuscripta Math.*, 114(1):61–116, 2004.
- [Bei84] A. A. Beĭlinson. Higher regulators and values of L -functions. In *Current problems in mathematics, Vol. 24, Itogi Nauki i Tekhniki*, pages 181–238. Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984.
- [Bei86] A. A. Beĭlinson. Notes on absolute Hodge cohomology. In *Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983)*, volume 55 of *Contemp. Math.*, pages 35–68. Amer. Math. Soc., Providence, RI, 1986.
- [Ber97a] Pierre Berthelot. Dualité de Poincaré et formule de Künneth en cohomologie rigide. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(5):493–498, 1997.
- [Ber97b] Pierre Berthelot. Finitude et pureté cohomologique en cohomologie rigide. *Invent. Math.*, 128(2):329–377, 1997. With an appendix in English by Aise Johan de Jong.
- [Bes00] Amnon Besser. Syntomic regulators and p -adic integration. I. Rigid syntomic regulators. In *Proceedings of the Conference on p -adic Aspects of the Theory of Automorphic Representations (Jerusalem, 1998)*, volume 120, pages 291–334, 2000.
- [Blo86] Spencer Bloch. Algebraic cycles and higher K -theory. *Adv. in Math.*, 61(3):267–304, 1986.
- [BLR95] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. Formal and rigid geometry. IV. The reduced fibre theorem. *Invent. Math.*, 119(2):361–398, 1995.
- [Con07] Brian Conrad. Deligne’s notes on Nagata compactifications. *J. Ramanujan Math. Soc.*, 22(3):205–257, 2007.
- [Del71] Pierre Deligne. Théorie de Hodge. II. *Inst. Hautes Etudes Sci. Publ. Math.*, (40):5–57, 1971.
- [DI87] Pierre Deligne and Luc Illusie. Relèvements modulo p^2 et décomposition du complexe de de Rham. *Invent. Math.*, 89(2):247–270, 1987.

- [FS02] Eric M. Friedlander and Andrei Suslin. The spectral sequence relating algebraic K -theory to motivic cohomology. *Ann. Sci. École Norm. Sup. (4)*, 35(6):773–875, 2002.
- [GM87] Henri Gillet and William Messing. Cycle classes and Riemann-Roch for crystalline cohomology. *Duke Math. J.*, 55(3):501–538, 1987.
- [EGA4.3] Alexander Grothendieck. Éléments de géométrie algébrique. IV. Etude locale des schémas et des morphismes de schémas. III. *Inst. Hautes Etudes Sci. Publ. Math.*, (28):255, 1966.
- [Gro66b] Alexander Grothendieck. On the de Rham cohomology of algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (29):95–103, 1966.
- [EGA4.4] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, (32):361, 1967.
- [Har75] Robin Hartshorne. On the De Rham cohomology of algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (45):5–99, 1975.
- [Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
- [Hub95] Annette Huber. *Mixed motives and their realization in derived categories*, volume 1604 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1995.
- [Ivo05] Florian Ivorra. Réalisation l -adique des motifs mixtes. *These de doctorat de l'Université Paris*, 6, 2005.
- [Jan90] Uwe Jannsen. *Mixed motives and algebraic K-theory*, volume 1400 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1990. With appendices by S. Bloch and C. Schoen.
- [Lev98] Marc Levine. *Mixed motives*, volume 57 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.
- [LS07] Bernard Le Stum. *Rigid cohomology*, volume 172 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2007.
- [MVW06] Carlo Mazza, Vladimir Voevodsky, and Charles Weibel. *Lecture notes on motivic cohomology*, volume 2 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI, 2006.
- [Pet03] Denis Petrequin. Classes de Chern et classes de cycles en cohomologie rigide. *Bull. Soc. Math. France*, 131(1):59–121, 2003.
- [PS08] Chris A. M. Peters and Joseph H. M. Steenbrink. *Mixed Hodge structures*, volume 52 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2008.
- [SGA6] *Théorie des intersections et théorème de Riemann-Roch*. Lecture Notes in Mathematics, Vol. 225. Springer-Verlag, Berlin, 1971. Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre.
- [SGA4.1] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*. Lecture Notes in Mathematics, Vol. 269. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.
- [SGA4.2] *Théorie des topos et cohomologie étale des schémas. Tome 2*. Lecture Notes in Mathematics, Vol. 270. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.
- [SV00] Andrei Suslin and Vladimir Voevodsky. Relative cycles and Chow sheaves. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pages 10–86. Princeton Univ. Press, Princeton, NJ, 2000.
- [vdPS95] M. van der Put and P. Schneider. Points and topologies in rigid geometry. *Math. Ann.*, 302(1):81–103, 1995.
- [Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.
- [Wł05] Jarosław Włodarczyk. Simple Hironaka resolution in characteristic zero. *J. Amer. Math. Soc.*, 18(4):779–822 (electronic), 2005.